Gaussian Elimination

Gaussian elimination is a method to solve systems of equations.

First, write the system of equations into an augmented matrix (3x3 with an extra column)

$$\begin{cases} x + 2y = 12 \\ 2x + y + z = 14 \\ y + 3z = 16 \end{cases}$$

x-value column, y-value column, z-value column

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 2 & 1 & 1 & 14 \\ 0 & 1 & 3 & 16 \end{array}\right]$$

constants column

Goal:

Get 0’s in the bottom left corner.

Original Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 4 \\ 1 & 1 & 3 & 7 \end{array}\right] \rightarrow$$

Goal Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

0’s underneath the main diagonal

We use **row operations** to get to the goal.

Row operation include:

1) Swap rows
2) Multiply by a scalar
3) Add or subtract rows
4) Add/subtract rows after multiplying by a scalar

Perform the following row operations on A.

$$A = \left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 3 & 7 & -2 & 3 \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Adding rows: $R_1 + R_2 \rightarrow R_2$

$$\begin{array}{rrrrrl} & -1 & 2 & -5 & 4 & (R_1) \\ + & 3 & 7 & -2 & 3 & (R_2) \\ \hline & 2 & 9 & -7 & 7 & \end{array}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 2 & 9 & -7 & 7 \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Mult. By scalar: $1/7R_2 \rightarrow R_2$

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ {}^{3}/_{7} & 1 & -{}^{2}/_{7} & {}^{3}/_{7} \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Perform the following row operations on A.

$$A = \left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 3 & 7 & -2 & 3 \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Row swap: $R_1 \leftrightarrow R_2$

$$\left[\begin{array}{ccc|c} 3 & 7 & -2 & 3 \\ -1 & 2 & -5 & 4 \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Combination: $-3R_3 + 2R_2 \rightarrow R_2$

24	-12	-3	21	($-3R_3$)
+ 6	14	-4	6	($2R_2$)
30	2	-7	27	

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 30 & 2 & -7 & 27 \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Perform the following row operations on A.

$$A = \left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 3 & 7 & -2 & 3 \\ -8 & 4 & 1 & -7 \end{array}\right]$$

Use each solution as the beginning of the next step.

$-8R_1 + R_3 \rightarrow R_3$

	8	-16	40	-32	($-8R_1$)
+	-8	4	1	-7	(R_3)
	0	-12	41	-39	

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 3 & 7 & -2 & 3 \\ 0 & -12 & 41 & -39 \end{array}\right]$$

$3R_1 + R_2 \rightarrow R_2$

	-3	6	-15	12	($3R_1$)
+	3	7	-2	3	(R_2)
	0	13	-17	15	

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 0 & 13 & -17 & 15 \\ 0 & -12 & 41 & -39 \end{array}\right]$$

$R_2 + R_3 \rightarrow R_3$

	0	13	-17	15	(R_2)
+	0	-12	41	-39	(R_3)
	0	1	24	-24	

$$\left[\begin{array}{ccc|c} -1 & 2 & -5 & 4 \\ 0 & 13 & -17 & 15 \\ 0 & 1 & 24 & 24 \end{array}\right]$$

Solve the system.

$x + 2y + z = 8$

$2x + y - z = 4$

$x + y + 3z = 7$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 8 \\ 2 & 1 & -1 & 4 \\ 1 & 1 & 3 & 7 \end{array}\right]$$

$-2R_1 + R_2 \rightarrow R_2$

	-2	-4	-2	-16	$(-2R_1)$
+	2	1	-1	4	(R_2)
	0	-3	-3	-12	

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 8 \\ 0 & -3 & -3 & -12 \\ 1 & 1 & 3 & 7 \end{array}\right]$$

$R_1 - R_3 \rightarrow R_3$

	1	2	1	8	(R_1)
+	-1	-1	-3	-7	$(-R_3)$
	0	1	-2	1	

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 8 \\ 0 & -3 & -3 & 12 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$R_2 + 3R_3 \rightarrow R_2$

	0	-3	-3	-12	(R_2)
+	0	3	-6	3	$(3R_3)$
	0	0	-9	-9	

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 8 \\ 0 & 0 & -9 & -9 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$R_2 \leftrightarrow 3R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

We have achieved our goal. Now we can solve the system. The numbers to the left of the dotted line are coefficients.
So, the bottom line is 0x + 0y – 9z = -9.
Or, -9z = -9.

This means that z = 1.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

Next use the middle row.

0x + 1y – 2z = 1

Substitute z = 1 that we just found to get:

y – 2 = 1

Meaning y = 3.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -9 & -9 \end{array}\right]$$

Lastly, use the top row.

1x + 2y + 1z = 8

Substitute z = 1 and y = 3:

x + 2(3) + 1 = 8

x + 6 + 1 = 8

x + 7 = 8

x = 1.

The solution to the system is (1, 3, 1).

Solve the system.

$3x - y + 4z = -10$

$-x + y + 2z = 6$

$2x - y + z = -8$

$$\left[\begin{array}{ccc:c} 3 & -1 & 4 & -10 \\ -1 & 1 & 2 & 6 \\ 2 & -1 & 1 & -8 \end{array}\right]$$

$2R_2 + R_3 \rightarrow R_3$

	-2	2	4	12	($2R_2$)
+	2	-1	1	-8	(R_3)
	0	1	5	4	

$$\left[\begin{array}{ccc:c} 3 & -1 & 4 & -10 \\ -1 & 1 & 2 & 6 \\ 0 & 1 & 5 & 4 \end{array}\right]$$

$R_1 + 3R_2 \rightarrow R_2$

	3	-1	4	-10	(R_1)
+	-3	3	6	18	($3R_2$)
	0	2	10	8	

$$\left[\begin{array}{ccc:c} 3 & -1 & 4 & -10 \\ 0 & 2 & 10 & 8 \\ 0 & 1 & 5 & 4 \end{array}\right]$$

$-2R_3 + R_2 \rightarrow R_2$

	0	-2	-10	-8	($-2R_2$)
+	0	2	10	8	(R_2)
	0	0	0	0	

$$\left[\begin{array}{ccc:c} 3 & -1 & 4 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 3 & -1 & 4 & -10 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 4 \end{array}\right]$$

The middle row is: 0x + 0y + 0z = 0
Or, 0 = 0.
This is a true statement, so there is an infinite number of solutions to this system.

Solve the system.

$4x + 4y + z = 24$

$2x - 4y + z = 0$

$5x - 4y - 5z = 12$

$$\left[\begin{array}{ccc|c} 4 & 4 & 1 & 24 \\ 2 & -4 & 1 & 0 \\ 5 & -4 & -5 & 12 \end{array}\right]$$

Variation: get zeroes in the top right corner.

$R_1 - R_2 \rightarrow R_1$

$$\begin{array}{rrrrrl} & 4 & 4 & 1 & 24 & (R_1) \\ + & -2 & 4 & -1 & 0 & (-R_2) \\ \hline & 2 & 8 & 0 & 24 & \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 0 & 24 \\ 2 & -4 & 1 & 0 \\ 5 & -4 & -5 & 12 \end{array}\right]$$

$5R_2 + R_3 \rightarrow R_2$

$$\begin{array}{rrrrrl} & 10 & -20 & 5 & 0 & (5R_2) \\ + & 5 & -4 & -5 & 12 & (R_3) \\ \hline & 15 & -24 & 0 & 12 & \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 0 & 24 \\ 15 & -24 & 0 & 12 \\ 5 & -4 & -5 & 12 \end{array}\right]$$

$\frac{1}{3}R_2 + R_1 \rightarrow R_1$

$$\begin{array}{rrrrrl} & 5 & -8 & 0 & 4 & (\frac{1}{3}R_2) \\ + & 2 & 8 & 0 & 24 & (R_1) \\ \hline & 7 & 0 & 0 & 28 & \end{array}$$

$$\left[\begin{array}{ccc|c} 7 & 0 & 0 & 28 \\ 15 & -24 & 0 & 12 \\ 5 & -4 & -5 & 12 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 7 & 0 & 0 & 28 \\ 15 & -24 & 0 & 12 \\ 5 & -4 & -5 & 12 \end{array}\right]$$

Solve the top equation:

7x = 28

x = 4

Solve the middle equation:

15x – 24y = 12

15(4) – 24y = 12

60 – 24y = 12

-24y = -48

y = 2

$$\left[\begin{array}{ccc:c} 7 & 0 & 0 & 28 \\ 15 & -24 & 0 & 12 \\ 5 & -4 & -5 & 12 \end{array}\right]$$

Solve the bottom equation:

5x – 4y – 5z = 12

5(4) – 4(2) – 5z = 12

20 – 8 – 5z = 12

12 – 5z = 12

-5z = 0

z = 0

The solution to the system is (4, 2, 0).