Inverse of a Matrix

An **identity matrix** is a square matrix with 1's on the diagonal and 0's everywhere else.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When you multiply a matrix with its **inverse** (A^{-1}), you will get the identity matrix.

$$A \times A^{-1} = I$$

$$A^{-1} \times A = I$$

Are the following matrices inverses?

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0.5 \\ 0 & 1 \end{bmatrix}$$

Multiply the matrices together.

$$\begin{bmatrix} -2 & 0 \\ -2 & 1 \end{bmatrix}$$

This is not the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so they are not inverses.

A matrix will have an inverse if:

1) It is a square matrix   AND
2) Its determinant is <u>not</u> zero.

A matrix with a determinant of zero is called **<u>singular</u>**; it has no inverse.

Inverse of a 2x2 Matrix

The inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is: $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Find the inverse of the matrix if it is defined.

$$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

det(A) = (4)(1) – (3)(2)
det(A) = -2

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ 1 & -2 \end{bmatrix}.$$

Find the inverse of $\begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$, if it is defined.

det(M) = (3)(-2) – (3)(2)
det(M) = -12

$$M^{-1} = \frac{1}{-12}\begin{bmatrix} -2 & -2 \\ -3 & 3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{-2}{-12} & \frac{-2}{-12} \\ \frac{-3}{-12} & \frac{3}{-12} \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} 1/6 & 1/6 \\ 1/4 & -1/4 \end{bmatrix}$$

To solve systems of equations with the inverse, you first write the **matrix equation**.

The matrix equation representing $\begin{cases} x + y = 8 \\ 2x + y = 1 \end{cases}$ is shown.

$$A \bullet X = B$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Coefficient matrix A Variable matrix X Constant matrix B

To solve $AX = B$, multiply both sides by the inverse A^{-1}.

$$A^{-1}AX = A^{-1}B$$

$$IX = A^{-1}B$$ *The product of A^{-1} and A is I.*

$$X = A^{-1}B$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 15 \end{bmatrix}$$

$x = -7$, $y = 15$

Write the matrix equation for the system and solve.

$$\begin{cases} 2x + 5y = 0 \\ 5x - 3y = 31 \end{cases}$$

$$\begin{matrix} A & X & = & B \end{matrix}$$

$$\begin{bmatrix} 2 & 5 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 31 \end{bmatrix}$$

Set up the matrix equation.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & -3 \end{bmatrix}^{-1} \times \begin{bmatrix} 0 \\ 31 \end{bmatrix}$$

Multiply by the inverse of A.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

x = 5 and y = -2

Encoding a message

Matrices can be used to send encrypted messages.

You will have an encrypting matrix (E), a message matrix (M) and the coded (*encrypted*) matrix (C).

Multiply the encrypting matrix by the message matrix to create the coded matrix.

The coded message is what is sent

EM = C

B E L T O N

$$\begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 & 12 & 20 & 15 & 14 \\ 20 & 9 & 7 & 5 & 18 & 19 \end{bmatrix} = \begin{bmatrix} -38 & -13 & -2 & 10 & -21 & -24 \\ -34 & -3 & 22 & 50 & 9 & 4 \end{bmatrix}$$

T I G E R S

Make up the encrypting matrix

Multiply by the inverse of E to decode the encrypted message.

$E^{-1}EM = E^{-1}C$

$M = E^{-1}C$

$$\begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} -38 & -13 & -2 & 10 & -21 & -24 \\ -34 & -3 & 22 & 50 & 9 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 12 & 20 & 15 & 14 \\ 20 & 9 & 7 & 5 & 18 & 19 \end{bmatrix}$$