# Calculus Section 10.3 Parametric Equations and Calculus

-Find the slope of a tangent in given by parametric equations
-Find the arc length of a curve given by a set of parametric equation

first point only

Homework: page 711 #'s 5 – 11 odd, 15, 29, 32, 45 – 47

If a smooth curve *C* is given by the equations x = f(t) and y = g(t), then the slope of *C* at the point (x , y) is

given by: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}, where\ \dfrac{dx}{dt} \neq 0$. The second derivative is: $\dfrac{d^2y}{dx^2} = \underbrace{\dfrac{d}{dx}\left[\dfrac{dy}{dx}\right]}_{\text{implicit diff.}} = \dfrac{d}{dt}\left[\dfrac{dy}{dx}\right]\underbrace{\left(\dfrac{dt}{dx}\right)}_{\text{Chain rule}} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

**Example)**

Given the parametric equations $x = 2\sqrt{t}$ and $y = 3t^2 - 2t$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

$y = 3t^2 - 2t$ $\qquad x = 2t^{1/2}$

$\dfrac{dy}{dt} = 6t - 2$ $\qquad \dfrac{dx}{dt} = t^{-1/2}$

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$

$\dfrac{dy}{dx} = \dfrac{6t-2}{t^{-1/2}}$

$\boxed{\dfrac{dy}{dx} = 6t^{3/2} - 2t^{1/2}}$

$\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[6t^{3/2} - 2t^{1/2}\right]}{dx/dt}$

$\dfrac{d^2y}{dx^2} = \dfrac{9t^{1/2} - t^{-1/2}}{t^{-1/2}} = \boxed{9t - 1}$

Implicit Diff. Version:

$\dfrac{d}{dx}\left[6t^{3/2} - 2t^{1/2}\right] \rightarrow 9t^{1/2}\dfrac{dt}{dx} - t^{-1/2}\dfrac{dt}{dx} \rightarrow \left(9t^{1/2} - t^{-1/2}\right)\dfrac{dt}{dx}$

$\dfrac{9t^{1/2} - t^{-1/2}}{dx/dt} \rightarrow \dfrac{9t^{1/2} - t^{-1/2}}{t^{-1/2}} \rightarrow \boxed{9t - 1}$

**Example)**

Given the parametric equations $x = 4\cos t$ and $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \dfrac{3\pi}{4}$.

$x = 4\cos t$ $\qquad y = 3\sin t$

$\dfrac{dx}{dt} = -4\sin t$ $\qquad \dfrac{dy}{dt} = 3\cos t$

$\dfrac{dx}{dt} @ \dfrac{3\pi}{4} = -4\left(\dfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$\dfrac{dy}{dt} @ \dfrac{3\pi}{4} = 3\left(\dfrac{-\sqrt{2}}{2}\right) = \dfrac{-3\sqrt{2}}{2}$

$\dfrac{dy}{dx} = \dfrac{\frac{-3\sqrt{2}}{2}}{-2\sqrt{2}} = \dfrac{3}{4}$

$x\left(\dfrac{3\pi}{4}\right) = 4\cos\left(\dfrac{3\pi}{4}\right) = 4\left(-\dfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y\left(\dfrac{3\pi}{4}\right) = 3\sin\left(\dfrac{3\pi}{4}\right) = 3\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\sqrt{2}}{2}$

$\boxed{y - \dfrac{3\sqrt{2}}{2} = \dfrac{3}{4}\left(x + 2\sqrt{2}\right)}$

**Example)**
Find an equation of the tangent line to the curve x = 2 – 3cosθ and y = 3 + 2sinθ at the point (-1, 3).

$x = 2 - 3\cos\theta$
$-1 = 2 - 3\cos\theta$
$-3 = -3\cos\theta$
$1 = \cos\theta$
$\theta = 0, 2\pi$

$y = 3 + 2\sin\theta$
$3 = 3 + 2\sin\theta$
$0 = 2\sin\theta$
$0 = \sin\theta$
$\theta = 0, 2\pi$

$\frac{dx}{d\theta} = 3\sin\theta$ @ $\theta = 0$ $\frac{dx}{d\theta} = 3\sin(0) = 0$

$\frac{dy}{d\theta} = 2\cos\theta$ @ $\theta = 0$ $\frac{dy}{d\theta} = 2\cos(0) = 2$

$\frac{dy}{dx} = \frac{2}{0}$ ← vertical tangent line

$\boxed{x = -1}$

**Example)**
Find all points of horizontal and vertical tangency given the parametric equations $x = t^2 + t,\ y = t^2 - 3t + 5$.

$\frac{dx}{dt} = 2t + 1$ $\frac{dy}{dt} = 2t - 3$

$\frac{dy}{dx} = \frac{2t-3}{2t+1}$

vertical tangent: $\frac{\#}{0} \rightarrow \frac{dx}{dt} = 0$

horizontal tangent: $\frac{0}{\#} \rightarrow \frac{dy}{dt} = 0$

vertical tangent

$\frac{dx}{dt} = 0$
$2t + 1 = 0$
$t = -1/2$
$x(-1/2) = (\frac{-1}{2})^2 + (-\frac{1}{2}) = \frac{-1}{4}$
$y(-1/2) = (\frac{-1}{2})^2 - 3(\frac{-1}{2}) + 5 = \frac{27}{4}$
$\boxed{\left(\frac{-1}{4}, \frac{27}{4}\right)}$

horizontal tangent

$\frac{dy}{dt} = 0$
$2t - 3 = 0$
$t = 3/2$
$x(3/2) = 15/4$
$y(3/2) = 11/4$
$\boxed{\left(\frac{15}{4}, \frac{11}{4}\right)}$

**Arc Length in Parametric Form**
If a smooth curve C is given by x = f(t) and y = g(t) such that C does not intersect itself on the interval (a, b), then the arc length of C over the interval is given by: $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

**Example)**
A particle moves along the smooth curve given by x = $t^2$ + 1 and y = $4t^3$ – 1. How far did the particle travel between t = 0 and t = 5?

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = 12t^2$

$s = \int_0^5 \sqrt{(2t)^2 + (12t^2)^2}\, dt \rightarrow s = \int_0^5 \sqrt{4t^2 + 144t^4}\, dt$

$\boxed{s = 500.815}$