Calculus Section 10.4 Polar Coordinates and Graphs

Rectangular coordinates are in the form (x, y).

Polar coordinates are in the form (r, θ).

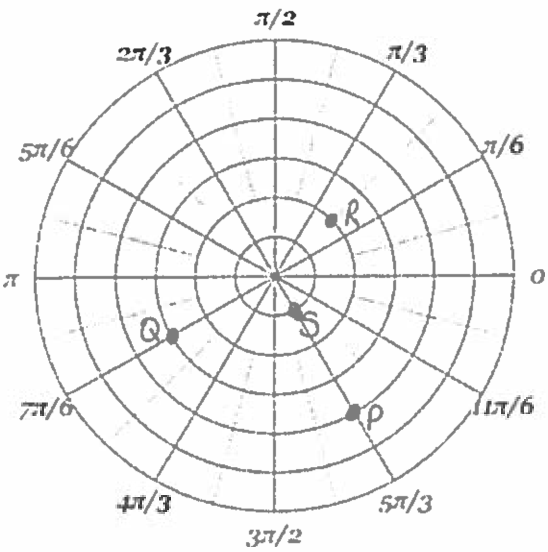
Ex. 1 Graph the following polar coordinates:

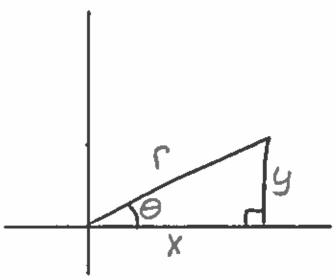
$P\left(4, \frac{5\pi}{3}\right)$

$Q\left(3, \frac{7\pi}{6}\right)$

$R\left(-2, \frac{5\pi}{4}\right)$

$S\left(1, -\frac{\pi}{3}\right)$





In Precalculus you learned that:

$\cos\theta = \frac{x}{r}$ so $x = r\cos\theta$

$\sin\theta = \frac{y}{r}$ so $y = r\sin\theta$

$\tan\theta = \frac{y}{x}$

$x^2 + y^2 = r^2$ so $r = \sqrt{x^2+y^2}$

Ex. Convert $\left(2, \frac{5\pi}{6}\right)$ to rectangular coordinates.

$x = r\cos\theta$

$x = 2\cos\left(\frac{5\pi}{6}\right)$

$x = 2\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{3}$

$y = r\sin\theta$

$y = 2\sin\left(\frac{5\pi}{6}\right)$

$y = 2\left(\frac{1}{2}\right) = 1$

$\boxed{(-\sqrt{3}, 1)}$

Ex. Convert $(3, -3)$ to polar coordinates.

$r = \sqrt{3^2 + (-3)^2}$

$r = \sqrt{9 + 9}$

$r = \sqrt{18} = 3\sqrt{2}$

$x = r\cos\theta$

$3 = 3\sqrt{2}\cos\theta$

$\frac{1}{\sqrt{2}} = \cos\theta$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

$\frac{7\pi}{4}$ is in the same quadrant as $(3, -3)$.

$\boxed{\left(3\sqrt{2}, \frac{7\pi}{4}\right)}$

Ex. Convert the following equations to polar form.

(a) $y = 4$

$4 = r\sin\theta$

$r = \frac{4}{\sin\theta}$

(b) $x^2 + y^2 = 25$

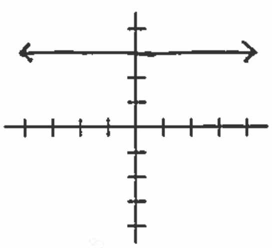
$r^2 = 25$

$r = 5$

Ex. Convert the following equations to rectangular form, and sketch the graph.

(a) $r\sin\theta = 3$

$y = 3$



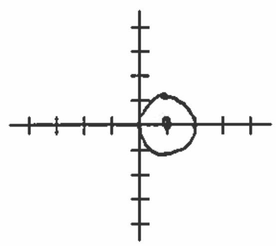
(b) $r = 2\cos\theta$

$r^2 = 2r\cos\theta$

$x^2 + y^2 = 2x$

complete the square $\rightarrow$ $x^2 - 2x + \underline{(-1)^2} + y^2 = \underline{(-1)^2}$

$(x-1)^2 + y^2 = 1$

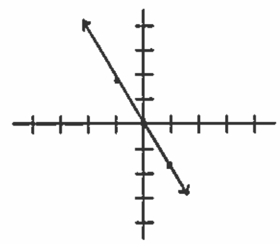


(c) $\theta = \frac{2\pi}{3}$

$\tan\theta = \tan\left(\frac{2\pi}{3}\right)$

$\frac{y}{x} = -\sqrt{3}$

$y = -\sqrt{3}\,x$



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To find the slope of a tangent line to a polar graph $r = f(\theta)$, we can use the facts that $x = r\cos\theta$ and $y = r\sin\theta$, together with the product rule:

$y = r\sin\theta$

$\frac{dy}{d\theta} = r\cos\theta + \sin\theta\frac{dr}{d\theta}$

$x = r\cos\theta$

$\frac{dx}{d\theta} = r(-\sin\theta) + \cos\theta\frac{dr}{d\theta}$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\cos\theta + \sin\theta\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Ex. Find $\frac{dy}{dx}$ and the slope of the graph of the polar curve at the given value of θ.

$r = 3 + 2\sin\theta, \quad \theta = \frac{\pi}{6}$

$\frac{dr}{d\theta} = 2\cos\theta$

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \qquad \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$$\frac{dy}{dx} = \frac{(3+2\sin\theta)(\cos\theta) + (\sin\theta)(2\cos\theta)}{-(3+2\sin\theta)(\sin\theta) + (\cos\theta)(2\cos\theta)}$$

$$\frac{dy}{dx} = \frac{(3+2(\frac{1}{2}))(\frac{\sqrt{3}}{2}) + (\frac{1}{2})(2\frac{\sqrt{3}}{2})}{-(3+2(\frac{1}{2}))(\frac{1}{2}) + (\frac{\sqrt{3}}{2})(2\frac{\sqrt{3}}{2})}$$

$$\rightarrow \frac{2\sqrt{3} + \frac{\sqrt{3}}{2}}{-2 + \frac{3}{2}} = \frac{\frac{5\sqrt{3}}{2}}{-\frac{1}{2}} = \boxed{-5\sqrt{3}}$$