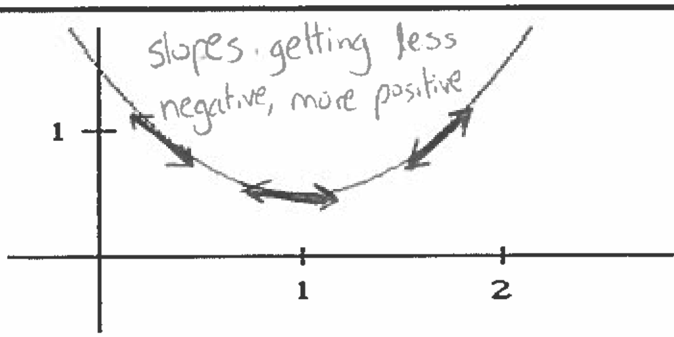
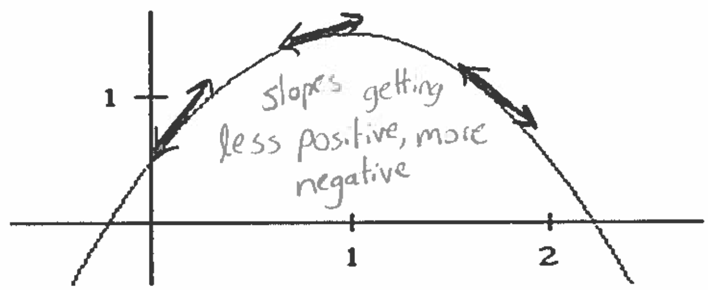
Calculus Section 3.4 Concavity and the Second Derivative Test

-Determine intervals on which a function is concave upward or concave downward
-Find any points of inflection of the graph of a function
-Apply the Second Derivative test to find relative extrema of a function

Homework: page 192 #'s 1, 2, 15, 17, 24, 33, 34, 50, 77, 78

Definition of Concavity
Let f be differentiable on an open interval. The graph of f is concave up if f' is increasing on the interval and concave down if f' is decreasing on the interval.





Test for Concavity
Let f be a function whose second derivative exists on an open interval I.
1) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward in I.
2) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward in I.

Example)
Determine the open intervals on which the graph of $f(x) = x^4 - 4x^3 + 2$ is concave upward or downward.

$f'(x) = 4x^3 - 12x^2$

$f''(x) = 12x^2 - 24x$

$0 = 12x(x-2)$

$x = 0 \quad x = 2$

x	-1	0	1	2	3
$f''(x)$	+	0	-	0	+

concave up: $(-\infty, 0) \cup (2, \infty)$

concave down: $(0, 2)$

Definition of Points of Inflection
Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f changes concavity at that point, then this point is a **point of inflection** of the graph of f.

Points of Inflection
If (c, f(c)) is a point of inflection of the graph of *f*, then either $f''(c) = 0$ or f'' does not exist at x = c.

Example: Finding points of inflection

Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^{\frac{2}{3}}(x-5)$.

$f(x) = x^{5/3} - 5x^{2/3}$

$f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3}$

$f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3}$

$0 = \frac{10}{9x^{1/3}} + \frac{10}{9x^{4/3}}$

$\frac{-10}{9x^{1/3}} = \frac{10}{9x^{4/3}}$ DNE at $x=0$

$-90x^{4/3} = 90x^{1/3}$

$-x^{4/3} = x^{1/3}$

$-x = 1$

$x = -1$

x	-2	-1	-1/2	0	1
f''(x)	-	0	+	DNE	+

concave up: $(-1,0) \cup (0,\infty)$

concave down: $(-\infty,-1)$

Point of inflection: $x = -1$

The Second Derivative Test
Let *f* be a function such that $f'(c) = 0$ and the second derivative of *f* exists on an open interval containing *c*.
1) If $f''(c) > 0$ (concave up), then *f* has a relative minimum at (c, f(c)).
2) If $f''(c) < 0$ (concave down), then *f* has a relative maximum at (c, f(c)).
3) If $f''(c) = 0$, then the test is inconclusive. It tells you NOTHING. You have to use the First Derivative Test instead. This does not mean that there is no max/min, the test just doesn't work for that function.

Example)
Use a tangent line at x = 1 to approximate f(1.1) for the function f(x) = $3x^2 + 2$. Tell whether the function is an overestimate or an underestimate. Justify your answer.

$f'(x) = 6x$

$f''(x) = 6$

$f'(1) = 6$ $\quad f(1) = 5$

$y - 5 = 6(x-1)$

$y = 6(1.1-1) + 5$

$y = 6(.1) + 5$

$y = 5.6$

$f(1.1) \approx 5.6$

$f(1.1) \approx 5.6$ is an overestimate because $f''(x) > 0$