Calculus Section 5.5 Exponential Functions w/ Function Bases

-Determine the derivative of exponential functions with functions as bases

Homework: page 362 #'s 63, 64, 67-70

When you have a function raised to a function power, use ln to differentiate.

Example)

Find $\frac{dy}{dx}$ for y = x^x

$y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{d}{dx}[\ln y = x \ln x]$

$\frac{1}{y}\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x(1)$

$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$

$\frac{dy}{dx} = (1 + \ln x)y$

$\boxed{\frac{dy}{dx} = (1 + \ln x)x^x}$

Example)

Find the equation of the line tangent to

$y = sinx^{cosx}$ at x = $\frac{\pi}{2}$.

$y = \sin x^{\cos x}$

$\ln y = \ln \sin x^{\cos x}$

$\ln y = \cos x \ln \sin x$

$\frac{d}{dx}[\ln y = \cos x \ln \sin x)$

$\frac{1}{y}\frac{dy}{dx} = \cos x\left(\frac{1}{\sin x}\right)\cos x + (\ln \sin x)(-\sin x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{\cos^2 x}{\sin x} - \sin x \ln \sin x$

$\frac{dy}{dx} = \left(\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\right)y$

$\frac{dy}{dx} = \left(\frac{\cos^2 x}{\sin x} - \sin x \ln \sin x\right)\sin x^{\cos x}$

@ $x = \pi/2$

$\frac{dy}{dx} = \left(\frac{0^2}{1} - 1\ln(1)\right)(1)^0$

$\frac{dy}{dx} = 0$

$y\left(\frac{\pi}{2}\right) = (1)^0 = 1$

$\boxed{y - 1 = 0\left(x - \frac{\pi}{2}\right) \text{ or } y = 1}$

A Special Limit Involving e

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x = e$$

Proof:

Let $y = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$\ln y = \ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)$

$\ln y = \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x$

$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$

$\ln y = \lim_{x\to\infty}\left(\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\right)$

Let $t=\frac{1}{x}$, $\lim_{x\to\infty} = \lim_{t\to 0^+}$

$\ln y = \lim_{t\to 0^+}\left(\frac{\ln(1+t)}{t}\right)$

$\ln 1 = 0$ so this does not change the expression

$\ln y = \lim_{t\to 0^+}\left(\frac{\ln(1+t)-\ln 1}{t}\right)$ ← limit definition of derivative of $\ln t$ at $t=1$

$\ln y = \frac{d}{dt}[\ln t]$

$\ln y = \frac{1}{t}$

at $t=1$

$\ln y = \frac{1}{1} = 1$

$\ln y = 1$

$e^{\ln y} = e^1$

$y = e$

$$\boxed{\text{Therefore, } \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e}$$