Calculus Section 5.7 Inverse Trig Integration

-Integrate functions whose antiderivatives involve inverse trig functions

Homework: page 380 #'s 1-8, 33

Let u be a differentiable function of x, and let a > 0

1) $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$

2) $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

3) $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

We only need these three because the pairs of derivatives from 5.6 match up with the negatives (i.e. an integral equaling arccosx is the same as -arcsinx, so arccosx isn't really needed).

Examples)

1) $\int \frac{dx}{\sqrt{4-x^2}}$ $\quad u=x \quad du=dx \quad a=2$

$\int \frac{du}{\sqrt{a^2-u^2}}$

$\arcsin\left(\frac{u}{a}\right) + C$

$\boxed{\arcsin \frac{x}{2} + C}$

2) $\int \frac{4}{2+9x^2}dx$ $\quad u=3x \quad du=3dx \quad \frac{1}{3}du=dx \quad a=\sqrt{2}$

$4 \times \frac{1}{3}\int \frac{du}{a^2+u^2}$

$\frac{4}{3} \times \frac{1}{a}\arctan\frac{u}{a} + C$

$\boxed{\frac{4}{3\sqrt{2}}\arctan\left(\frac{3x}{\sqrt{2}}\right) + C}$

3) $\int \frac{dx}{x\sqrt{4x^2-9}}$ $\quad u=2x \quad x=\frac{1}{2}u \quad du=2dx \quad \frac{1}{2}du=dx \quad a=3$

$\frac{1}{2}\int \frac{du}{\frac{1}{2}u\sqrt{u^2-a^2}}$

$2\cdot\frac{1}{2}\int \frac{du}{u\sqrt{u^2-a^2}}$

$\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$

$\boxed{\frac{1}{3}\operatorname{arcsec}\frac{2x}{3} + C}$

Integration by Substitution

Example)

$\int \frac{dx}{\sqrt{e^{2x}-1}}$ $\quad u=e^x \quad du=e^x dx \quad \frac{1}{e^x}du=dx \quad a=1$

$\int \frac{\frac{1}{e^x}du}{\sqrt{u^2-a^2}}$

$\int \frac{du}{e^x\sqrt{u^2-a^2}}$

$\int \frac{du}{u\sqrt{u^2-a^2}}$

$\frac{1}{a}\operatorname{arcsec}\frac{u}{a} + C$

$\boxed{\operatorname{arcsec}(e^x) + C}$

$\int \frac{x+2}{\sqrt{4-x^2}}dx$

$\int \frac{x}{\sqrt{4-x^2}}dx + 2\int \frac{dx}{\sqrt{4-x^2}}$

$u=4-x^2 \quad du=-2x\,dx \quad -\frac{1}{2}du=x\,dx$

$v=x \quad dv=dx \quad a=2$

$-\frac{1}{2}\int u^{-1/2}du + 2\int \frac{dv}{\sqrt{a^2-v^2}}$

$-\frac{1}{2}\cdot 2u^{1/2} + 2\arcsin\frac{v}{a} + C$

$\boxed{-\sqrt{4-x^2} + 2\arcsin\frac{x}{2} + C}$

$$\int \frac{x-2}{(x+1)^2+4}dx \qquad X-2=(X+1)-3$$

$$\int \frac{(x+1)-3}{(x+1)^2+4}dx$$

$$\int \frac{x+1}{(x+1)^2+4}dx - 3\int \frac{dx}{(x+1)^2+4}$$

$u=(x+1)^2+4$ $\qquad$ $v=x+1 \quad a=2$

$du=2(x+1)dx$ $\qquad$ $dv=dx$

$\frac{1}{2}du=(x+1)dx$

$$\frac{1}{2}\int \frac{1}{u}du - 3\int \frac{dv}{v^2+a^2}$$

$$\frac{1}{2}\ln|u| - 3\times\frac{1}{a}\arctan\frac{v}{a}+C$$

$$\boxed{\frac{1}{2}\ln|(x+1)^2+4| - \frac{3}{2}\arctan\left(\frac{x+1}{2}\right)+C}$$

Completing the Square

Find $\int \frac{dx}{x^2-4x+7}$

$$\left(x^2-4x+\left(\frac{-2}{}\right)^2\right)+7-\left(\frac{-2}{}\right)^2$$

$$(x-2)^2+7-4$$

$$(x-2)^2+3$$

$$\int \frac{dx}{(x-2)^2+3} \qquad u=x-2 \quad a=\sqrt{3} \qquad du=dx$$

$$\int \frac{du}{u^2+a^2}$$

$$\frac{1}{a}\arctan\frac{u}{a}+C$$

$$\boxed{\frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right)+C}$$