Calculus Section 6.2 Growth and Decay

Homework: page 412 #'s 3, 5, 15, 19, 20, 51, 52, 56, 65

-Use exponential functions to model growth and decay in applied problems

In many applications, the rate of change of a variable is proportional to the value of y. In these problems, k represents the <u>proportionality constant</u>. Two general types of variation are direct and indirect variation. If y is a function of time *t*, the proportions can be written as follows:

Dircect $\frac{dy}{dt} = ky$

$\int \frac{1}{y} dt = \int k\,dt$

$\ln|y| = kt + C$

$e^{\ln|y|} = e^{kt+C}$

$y = Ce^{kt}$

Indirect: $\frac{dy}{dt} = \frac{k}{y}$

$\int y\,dy = \int k\,dt$

$\frac{1}{2}y^2 = kt + C$

$y^2 = 2kt + C$

Example) Using an exponential growth model

The rate of change of y is proportional to y. When t = 0, y = 2. When t = 2, y = 4. What is the value of y when t = 3?

$\frac{dy}{dt} = ky$

$y = Ce^{kt}$

$2 = Ce^{k(0)}$

$2 = Ce^{0}$

$2 = C$

$y = 2e^{kt}$

$4 = 2e^{k(2)}$

$2 = e^{2k}$

$\ln 2 = 2k$

$\frac{1}{2}\ln 2 = k$

$y = 2e^{(\frac{1}{2}\ln 2)t}$

$y = 2e^{(\frac{1}{2}\ln 2)(3)}$

$\boxed{y = 5.657}$

Example) Radioactive decay

The half-life of Plutonium-239 is 24,100 years. Suppose that 10 grams of Plutonium-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

$y = Ce^{kt}$

$y = 10e^{kt}$

$5 = 10e^{k(24100)}$

$\frac{1}{2} = e^{24100k}$

$\ln(\frac{1}{2}) = 24100k$

$\frac{\ln(\frac{1}{2})}{24100} = k$

$y = 10e^{\frac{\ln(1/2)}{24100}t}$

$1 = 10e^{\frac{\ln(1/2)}{24100}t}$

$.1 = e^{\frac{\ln(1/2)}{24100}t}$

$\ln(.1) = \frac{\ln(1/2)}{24100}t$

$t = \frac{\ln(.1)}{\ln(1/2)/24100} = \boxed{80{,}058.467 \text{ years}}$

Example) When you don't start with a value at t = 0

Suppose an experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

$y = Ce^{kt}$

$100 = Ce^{2k}$ $\quad 300 = Ce^{4k}$

$C = \frac{100}{e^{2k}}$

$300 = \frac{100}{e^{2k}} e^{4k}$

$300 = 100e^{2k}$

$3 = e^{2k}$

$\ln 3 = 2k \rightarrow k = \frac{1}{2}\ln 3$

$C = \frac{100}{e^{2(\frac{1}{2}\ln 3)}}$

$C = 33.\overline{3}$

33 flies in the original population

Example) Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. The object cools from 100° to 90° in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°? Newton's Law of Cooling: $\frac{dT}{dt} = k(T - M)$ where M is the surrounding temperature and T is the temperature of the object.

$\frac{dT}{dt} = k(T-M)$

$\frac{dT}{dt} = k(T-60)$

$\int \frac{1}{T-60}\, dT = \int k\, dt$

$\ln|T-60| = kt + C$

$T - 60 = Ce^{kt}$

$T = 60 + Ce^{kt}$

$100 = 60 + Ce^{k(0)}$

$40 = C$

$T = 60 + 40e^{kt}$

$90 = 60 + 40e^{10k}$

$30 = 40e^{10k}$

$\frac{3}{4} = e^{10k}$

$\ln(3/4) = 10k$

$\frac{1}{10}\ln(3/4) = k$

$T = 60 + 40e^{\frac{1}{10}\ln(3/4)t}$

$80 = 60 + 40e^{\frac{1}{10}\ln(3/4)t}$

$20 = 40e^{\frac{1}{10}\ln(3/4)t}$

$\frac{1}{2} = e^{\frac{1}{10}\ln(3/4)t}$

$\ln(\frac{1}{2}) = \frac{1}{10}\ln(3/4)t$

$t = \frac{10\ln(1/2)}{\ln(3/4)}$

$t = 24.094$

It takes 14.094 minutes to cool from 90° to 80°.