Calculus Section 6.3 Logistic Growth

In exponential growth, we assume that the rate of increase (or decrease) of a population at any time t is directly proportional to the population P. In other words, $\frac{dP}{dt} = kP$.. However, in many situations population growth levels off and approaches a limiting number L (the carrying capacity) because of limited resources. In this situation the rate of increase (or decrease) is directly proportional to both P and $L-P$. This type of growth is called **logistic growth**. It is modeled by the differential equation $\frac{dP}{dt} = kP(L-P)$.

If we find $\frac{d^2P}{dt^2}$, we can find out an important fact about the time when P is growing the fastest.

Example) The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$, where t is measured in years, and $P(0) = 4000$.

$\frac{dP}{dt} = \frac{1}{6000}P(18000-P)$

(a) $\lim_{t\to\infty} P(t) =$ 18000

(b) What is the range of the solution curve?

$4000 \leq P \leq 18000$

(c) For what values of P is the solution curve increasing? Decreasing? Justify your answer.

$\frac{dP}{dt} = 0$

$\frac{1}{6000}P(18000-P) = 0$

$P = 0 \quad P = 18000$

P	0	9000	18000	20000
$\frac{dP}{dt}$	0	+	0	−

$(0, 18000)$ increasing

$(18000, \infty)$ decreasing

(d) For what values of P is the solution curve concave up? Concave down? Justify your answer.

$\frac{dP}{dt} = 3P - \frac{P^2}{6000}$

$\frac{d^2P}{dt^2} = 3\frac{dP}{dt} - \frac{2P\frac{dP}{dt}}{6000}$

$\frac{d^2P}{dt^2} = \frac{dP}{dt}\left(3 - \frac{P}{3000}\right)$

$\frac{d^2P}{dt^2} = \frac{1}{3000}\frac{dP}{dt}(9000 - P)$

$\frac{d^2P}{dt^2} = 0$ when $P = 9000, 0, 18000$

P	0	5000	9000	12000	18000	2000
$\frac{d^2P}{dt^2}$	0	+	0	−	0	+

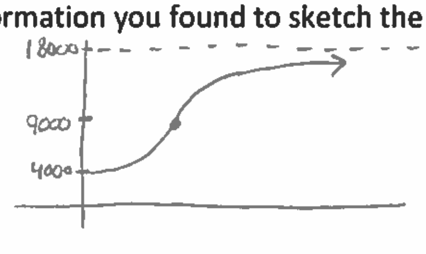
concave up: $(0, 9000) \cup (18000, \infty)$

concave down: $(9000, 18000)$

(e) Does the solution curve have an inflection point? Justify your answer.

yes, at $P = 9000$ $\frac{d^2P}{dt^2}$ changes signs

(f) Use the information you found to sketch the graph of $P(t)$.



Example) The population $P(t)$ of fish in a lake satisfies the logistic differential equation $\frac{dP}{dt} = 3P - \frac{P^2}{6000}$, where t is measured in years, and $P(0) = 10{,}000$.

$\frac{dP}{dt} = \frac{1}{6000} P(18000 - P)$

(a) $\lim_{t \to \infty} P(t) = 18000$ (b) What is the range of the solution curve?

$10{,}000 \le P \le 18{,}000$

(c) For what values of *P* is the solution curve increasing? Decreasing? Justify your answer.

$\frac{dP}{dt} = 0$ increasing: $(10000, 18000)$

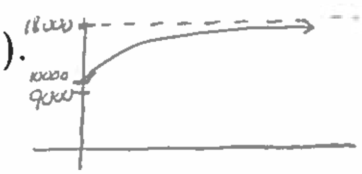
(d) For what values of *P* is the solution curve concave up? Concave down? Justify your answer.

$\frac{d^2P}{dt^2} = 0$ concave up: $(10000, 18000)$

(e) Does the solution curve have an inflection point? Justify your answer.

No, $\frac{d^2P}{dt^2}$ does not change signs

(f) Use the information you found to sketch the graph of $P(t)$.



Ex. 3 The population $P(t)$ of fish in a lake satisfies the logistic differential equation

$$\frac{dP}{dt} = 3P - \frac{P^2}{6000}, \text{ where } t \text{ is measured in years, and } P(0) = 20{,}000.$$

$\frac{dP}{dt} = \frac{1}{6000} P(18000 - P)$

(a) $\lim_{t \to \infty} P(t) = 18000$ (b) What is the range of the solution curve? $18000 \le P \le 20000$

(c) For what values of *P* is the solution curve increasing? Decreasing? Justify your answer.

$\frac{dP}{dt} = 0$ decreasing: $(18000, 20000)$

(d) For what values of *P* is the solution curve concave up? Concave down? Justify your answer.

$\frac{d^2P}{dt^2} = 0$ concave up: $(18000, 20000)$

(e) Does the solution curve have an inflection point? Justify your answer.

No, $\frac{d^2P}{dt^2}$ does not change signs

(f) Use the information you found to sketch the graph of $P(t)$.