Calculus Section 6.3 Separation of Variables

-Use separation of variables to solve simple differential equations

Homework: page 421 #'s 1-9 odd, 15, 17, 23, 25

Every day, physical phenomena can be represented and described by differential equations. Examples include population growth, sales predictions, radioactive decay, and Newton's Law of Cooling.

A function is a solution of a differential equation if the differential equation is satisfied by substituting the solution into the equation. For example, $y = e^{-2x}$ is a solution to the differential equation $y' + 2y = 0$.

$y = e^{-2x}$

$y' = -2e^{-2x}$

$-2e^{-2x} + 2(e^{-2x}) = 0$

$0 = 0$

$y = Ce^{-2x}$ is called a general solution to the differential equation because it has an arbitrary constant yet still solves the differential equation. Solutions to differential equations may not be unique.

Differential equations may be solved by using a process called separation of variables. To solve, separate the variables to opposite sides of the equal sign and integrate.

Example)

Solve the differential equation $y' = 2x/y$

$\frac{dy}{dx} = \frac{2x}{y}$

$\int y\,dy = \int 2x\,dx$

$\frac{1}{2}y^2 = x^2 + C$

$\boxed{y^2 = 2x^2 + C}$

Example)

Solve the differential equation $(x^2+4)\frac{dy}{dx} = xy$

$\int \frac{1}{y}\,dy = \int \frac{x}{x^2+4}\,dx$

$u = x^2+4$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{1}{u}\,du$

$\ln|y| = \frac{1}{2}\ln|x^2+4| + C$

$\ln|y| = \ln\sqrt{x^2+4} + C$

$e^{\ln|y|} = e^{\ln\sqrt{x^2+4} + C}$

$|y| = e^{\ln\sqrt{x^2+4}} \cdot e^C$

$|y| = \sqrt{x^2+4} \cdot C$

$|y| = C\sqrt{x^2+4}$

$y = \pm C\sqrt{x^2+4}$

$\boxed{y = C\sqrt{x^2+4}}$

Example) Find the particular solution

Given the initial condition y(0) = 1, find the particular solution of the equation $xydx + e^{-x^2}(y^2 - 1)dy = 0$.

$e^{-x^2}(y^2-1)dy = -xydx$

$\frac{y^2-1}{y}\,dy = \frac{-x}{e^{-x^2}}\,dx$

$\int \left(y - \frac{1}{y}\right) dy = \int -xe^{x^2}\,dx$ $\quad u = x^2$, $du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

$\frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + C$

$\boxed{\frac{1}{2}y^2 - \ln|y| = -\frac{1}{2}e^{x^2} + 1}$

$\frac{1}{2}(1)^2 - \ln|1| = -\frac{1}{2}e^{0^2} + C$

$\frac{1}{2} - 0 = -\frac{1}{2} + C$

$1 = C$

Example) Find a particular solution curve

Find the equation of the curve that passes through the point (1, 3) and has a slope of y/x^2 at any point (x, y).

slope $= \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y}{x^2}$

$\int \frac{1}{y}\,dy = \int x^{-2}\,dx$

$\ln|y| = -x^{-1} + C$

$\ln|y| = \frac{-1}{x} + C$

$e^{\ln|y|} = e^{-\frac{1}{x} + C}$

$|y| = e^{-1/x} \cdot e^{C}$

$|y| = Ce^{-1/x}$

$y = Ce^{-1/x}$

$3 = Ce^{-1/1}$

$3 = Ce^{-1}$

$3 = \frac{C}{e}$

$3e = C$

$y = (3e)e^{-1/x}$

$\boxed{y = 3e^{-1/x + 1}}$