Calculus Section 7.4 Arc Length and Surfaces of Revolution

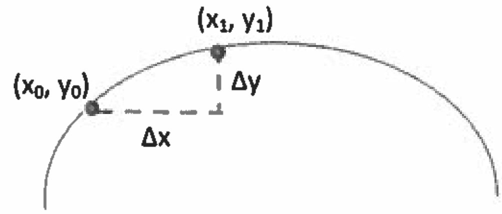
-Find the length of an arc

-Find the area of a surface of revolution

Homework: page 473 #'s 3, 11, 13, 21, 38, 43, 44, 47

Arc Length

The length of a curve is found by summing small segments using distance formula.



$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

$d = \sqrt{(\Delta x)^2 + (\Delta y)^2\left(\frac{\Delta x}{\Delta x}\right)^2}$ multiply by "1"

$d = \sqrt{(\Delta x)^2 + \left(\frac{\Delta y}{\Delta x}\right)^2(\Delta x)^2}$ pull Δx in with Δy

$d = \sqrt{(\Delta x)^2\left(1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right)}$ factor Δx^2

$d = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}(\Delta x)$ take square root Δx^2 to bring out of $\sqrt{}$

$d = \sqrt{1 + f'(x)^2}(\Delta x)$ $\frac{\Delta y}{\Delta x}$ is the slope, slope is $f'(x)$

$length = \int_a^b \sqrt{1 + f'(x)^2}dx$ if Δx is infinitely small, take an integral to add up all the little pieces

Examples)

Find the arc length of $f(x) = x^2$ on the interval [-1, 1].

$f(x) = x^2$

$f'(x) = 2x$

$length = \int_{-1}^{1} \sqrt{1 + (2x)^2}\,dx$

$length = 2.958$

Find the arc length of $y = \frac{3}{2}x^{2/3}$ on the interval [1, 8].

$y = \frac{3}{2}x^{2/3}$

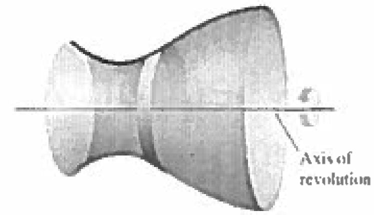
$y' = x^{-1/3}$

$length = \int_1^8 \sqrt{1 + (x^{-1/3})^2}\,dx$

$length = 8.352$

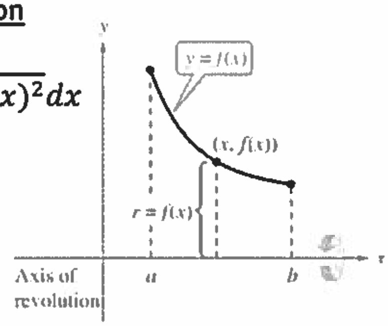
Surface of Revolution

We can combine the concepts of volume by rotation and arc length to find the surface area of a solid of revolution.



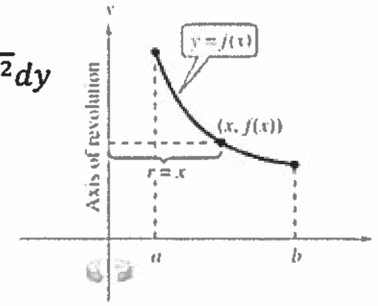
Horizontal axis of rotation

$$S = 2\pi \int_a^b r(x)\sqrt{1 + f'(x)^2}\,dx$$



Vertical axis of rotation

$$S = 2\pi \int_c^d r(y)\sqrt{1 + g'(y)^2}\,dy$$



where r(x) and r(y) are perpendicular to the axis of revolution.

Example)

Find the area of the surface of revolution formed by revolving the graph of $f(x) = x^3$ on the interval [0, 1] about the x-axis.

$f(x) = x^3$

$f'(x) = 3x^2$

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2}\,dx$$

$$S = 3.563$$

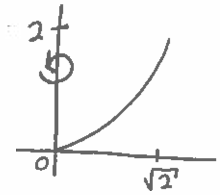
Example)

Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}\,]$ about the y-axis.

$y = x^2 \rightarrow x = \sqrt{y}$

$x' = \frac{1}{2}y^{-1/2}$

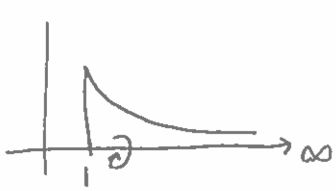
$$S = 2\pi \int_0^2 \sqrt{y}\,\sqrt{1 + (\tfrac{1}{2}y^{-1/2})^2}\,dy$$



$$S = 13.614$$

Gabriel's Horn

Gabriel's Horn is the name given to the 3-d figure created when the function $f(x) = \frac{1}{x}$ on the interval $[1, \infty]$ is revolved about the x-axis. Gabriel's Horn is a paradox because it has finite volume yet infinite surface area.



$f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx$$

$$V = \pi \int_1^\infty x^{-2}\,dx$$

$$V = \pi \left[-x^{-1}\right]_1^\infty$$

$$V = \pi\left[-\frac{1}{\infty} - \frac{-1}{1}\right]$$

$$V = \pi(0 + 1)$$

$$V = \pi$$

$$S = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1 + (-x^{-2})^2}\,dx$$

$$\int_1^\infty \frac{1}{x} = \ln|x|\Big|_1^\infty = \ln\infty - \ln 1 = \infty$$

$$\sqrt{1 + \frac{1}{x^2}} \geq \frac{1}{x}, \text{ so } S = \infty$$