Calculus Section 8.3 Trig Functions with Powers

-Solve trig integrals involving powers of sine and cosine

Homework: page 530 #'s 1, 3, 5, 9, 21, 25, 27

In this section we will evaluate integrals of the form $\int \sin^m x \cos^n x dx$ and $\int \sec^m x \tan^n x dx$ where either m or n is a positive integer. We will re-write the integrand as a combination of trig functions so the Power Rule may be used.

To break up the integral into manageable parts, use the following identities:

$sin^2x + cos^2x = 1$ Pythagorean Identity

$sin^2(ax) = \frac{1-\cos(2ax)}{2}$ Half-angle identify for $\sin^2 x$

$cos^2(ax) = \frac{1+\cos(2ax)}{2}$ Half-angle identify for $\cos^2 x$

Guidelines for Evaluating Integrals Involving Sine and Cosine

1) If the power of sine is odd and positive, save one sine and convert the rest to cosines.
2) If the power of cosine is odd and positive, save one cosine and convert the rest to sines.
3) If the powers of both the sine and cosine are even and nonnegative, use the half-angle identities to convert the integrand to odd powers of the cosine.

Exception: If one of the powers is 1, try simple u-substitution.

Example) Power of Sine is Odd and Positive

Find $\int \sin^3 x \cos^4 x dx$

$\int \sin x (\sin^2 x) \cos^4 x \, dx$

$\int \sin x (1 - \cos^2 x) \cos^4 x \, dx$

$\int \sin x (\cos^4 x - \cos^6 x) \, dx$

$\int \cos^4 x \sin x \, dx - \int \cos^6 x \sin x \, dx$

$u = \cos x \qquad du = -\sin x \, dx$

$-\int u^4 du + \int u^6 du$

$-\frac{1}{5}u^5 + \frac{1}{7}u^7$

$\boxed{-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}$

Example) Power of Cosine is Odd and Positive

Find $\int_0^{\pi/2} \frac{\cos^3 x}{\sqrt{\sin x}} dx$

$\int_0^{\pi/2} \cos x (\cos^2 x) \sin^{-1/2} x \, dx$

$\int_0^{\pi/2} \cos x (1 - \sin^2 x) \sin^{-1/2} x \, dx$

$\int_0^{\pi/2} \sin^{-1/2} x \cos x \, dx - \int_0^{\pi/2} \sin^{3/2} x \cos x \, dx$

$u = \sin x \qquad du = \cos x \, dx \qquad u(\pi/2) = 1$

$u(0) = 0$

$\int_0^1 u^{-1/2} du - \int_0^1 u^{3/2} du$

$\left[2u^{1/2} - \frac{2}{5}u^{5/2}\right]_0^1$

$(2 - \frac{2}{5}) - (0 - 0)$

$\boxed{8/5}$

Example) Power is Even and Nonnegative

Find $\int cos^4(5x)\,dx$

$\int (\cos^2(5x))^2 dx$

$\int \left(\frac{1+\cos(10x)}{2}\right)^2 dx$

$\int \left(\frac{1}{2} + \frac{1}{2}\cos(10x)\right)^2 dx$

$\int \left(\frac{1}{4} + \frac{1}{2}\cos(10x) + \frac{1}{4}\cos^2(10x)\right) dx$

$\int \left(\frac{1}{4} + \frac{1}{2}\cos(10x) + \frac{1}{4}\left(\frac{1+\cos(20x)}{2}\right)\right) dx$

$\int \left(\frac{1}{4} + \frac{1}{2}\cos(10x) + \frac{1}{8} + \frac{1}{8}\cos(20x)\right) dx$

$\frac{1}{4}x + \frac{1}{20}\sin(10x) + \frac{1}{8}x + \frac{1}{160}\sin(20x) + C$

$\boxed{\frac{3}{8}x + \frac{1}{20}\sin(10x) + \frac{1}{160}\sin(20x) + C}$

Guidelines for Evaluating Integrals Involving Secant and Tangent (Note: $1+\tan^2 x = \sec^2 x$)

1) If the power of secant is even and positive, save a secant-squared factor and convert the rest to tangents.
2) If the power of the tangent is odd and positive, save a secant-tangent and convert the rest to secants.
3) If there are no secants and the power of tangent is even and positive, convert a $\tan^2$ into $\sec^2 - 1$. Expand and repeat as necessary.
4) If the integral is only secant with an odd positive power, use integration by parts.
5) If none of the first four guidelines apply, try to convert to sines and cosines.

Example) Power of Tangent is Odd and Positive

Find $\int \frac{\tan^3 x}{\sqrt{\sec x}} dx$

$\int \tan^3 x \sec^{-1/2} x\, dx$

$\int \tan^3 x \sec x \sec^{-3/2} x\, dx$

$\int \sec x \tan x (\tan^2 x) \sec^{-3/2} x\, dx$

$\int \sec x \tan x (\sec^2 x - 1) \sec^{-3/2} x\, dx$

$\int \sec^{1/2} x \sec x \tan x\, dx - \int \sec^{-3/2} x \sec x \tan x\, dx$

$u = \sec x \qquad du = \sec x \tan x\, dx$

$\int u^{1/2} du - \int u^{-3/2} du$

$\frac{2}{3}u^{3/2} + 2u^{-1/2}$

$\boxed{\frac{2}{3}\sec^{3/2} x + \frac{2}{\sqrt{\sec x}} + C}$

Example) Power of Secant is Even and Positive

Find $\int \sec^4 3x \tan^3 3x\, dx$

$\int \sec^2 3x (\sec^2 3x) \tan^3 3x\, dx$

$\int \sec^2 3x (1 + \tan^2 3x) \tan^3 3x\, dx$

$\int \sec^2 3x (\tan^3 3x + \tan^5 3x)\, dx$

$\int \tan^3 3x \sec^2 3x\, dx + \int \tan^5 3x \sec^2 3x\, dx$

$u = \tan 3x \qquad du = 3\sec^2 3x\, dx$

$\frac{1}{3}\int u^3 du + \frac{1}{3}\int u^5 du$

$\frac{1}{12}u^4 + \frac{1}{18}u^6$

$\boxed{\frac{1}{12}\tan^4 3x + \frac{1}{18}\tan^6 3x + C}$