Calculus Section 8.4 Trig Substitution

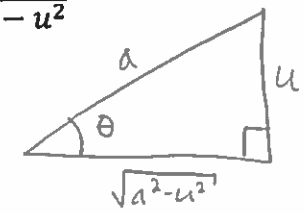
-Use trig substitution to solve integrals

Homework: page 539 #'s 24, 25, 30

Trig substitution is used to solve integrals resembling $\sqrt{a^2-u^2}$, $\sqrt{a^2+u^2}$, and $\sqrt{u^2-a^2}$. The idea is to eliminate the radical from the integral.

The setups:

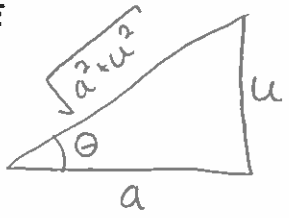
$\sqrt{a^2-u^2}$



$\sin\theta = \frac{u}{a}$ | $\cos\theta = \frac{\sqrt{a^2-u^2}}{a}$

$u = a\sin\theta$ | $\sqrt{u^2-a^2} = a\cos\theta$

$\sqrt{a^2+u^2}$

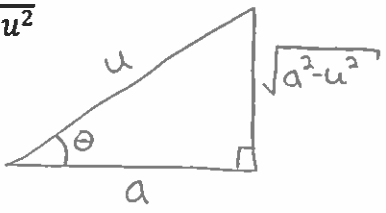


$\tan\theta = \frac{u}{a}$ | $\cos\theta = \frac{a}{\sqrt{a^2+u^2}}$

$u = a\tan\theta$ | $\sqrt{a^2+u^2} = \frac{a}{\cos\theta}$

$\sqrt{a^2+u^2} = a\sec\theta$

$\sqrt{a^2-u^2}$



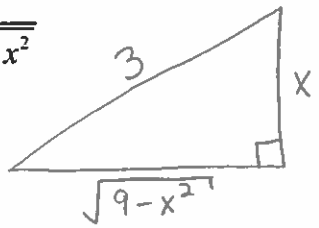
$\cos\theta = \frac{a}{u}$ | $\tan\theta = \frac{\sqrt{a^2-u^2}}{a}$

$u = \frac{a}{\cos\theta}$ | $\sqrt{a^2-u^2} = a\tan\theta$

$u = a\sec\theta$

Examples)

Find $\int \frac{dx}{x^2\sqrt{9-x^2}}$



$x = 3\sin\theta$ $\quad \sqrt{9-x^2} = 3\cos\theta$

$dx = 3\cos\theta d\theta$

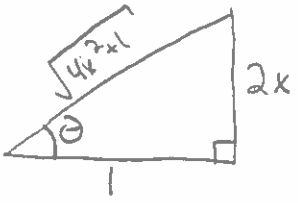
$$\int \frac{3\cos\theta}{(3\sin\theta)^2(3\cos\theta)}\,d\theta$$

$$\int \frac{1}{9\sin^2\theta}\,d\theta$$

$$\frac{1}{9}\int \csc^2\theta\,d\theta \rightarrow \frac{-1}{9}\cot\theta$$

$$\boxed{-\frac{1}{9}\left(\frac{\sqrt{9-x^2}}{x}\right)+C}$$

Find $\int \frac{dx}{\sqrt{4x^2+1}}$



$2x = \tan\theta$

$2dx = \sec^2\theta\,d\theta$

$dx = \frac{1}{2}\sec^2\theta\,d\theta$ $\quad \sqrt{4x^2+1} = \sec\theta$

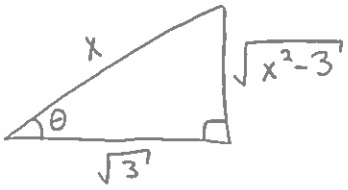
$$\int \frac{\frac{1}{2}\sec^2\theta}{\sec\theta}\,d\theta$$

$$\frac{1}{2}\int \sec\theta\,d\theta$$

$$\frac{1}{2}\ln|\sec\theta+\tan\theta|$$

$$\boxed{\frac{1}{2}\ln\left|\sqrt{4x^2+1}+2x\right|+C}$$

Find $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x}\,dx$



$x = \sqrt{3}\sec\theta$

$dx = \sqrt{3}\sec\theta\tan\theta$

$\sqrt{x^2-3} = \sqrt{3}\tan\theta$

$$\int_0^{\pi/6} \frac{\sqrt{3}\tan\theta}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta\,d\theta$$

$$\int_0^{\pi/6} \sqrt{3}\tan^2\theta\,d\theta$$

$$\sqrt{3}\int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta$$

$$\sqrt{3}\int_0^{\pi/6} \sec^2\theta\,d\theta - \sqrt{3}\int_0^{\pi/6} d\theta$$

$$\sqrt{3}\tan\theta - \sqrt{3}\theta\Big|_0^{\pi/6}$$

when $x = \sqrt{3}$

$\sqrt{3} = \sqrt{3}\sec\theta$

$1 = \sec\theta$

$1 = \frac{1}{\cos\theta}$

$\theta = 0$

when $x = 2$

$2 = \sqrt{3}\sec\theta$

$\frac{2}{\sqrt{3}} = \frac{1}{\cos\theta}$

$\cos\theta = \frac{\sqrt{3}}{2}$

$\theta = \pi/6$

$$\left(\sqrt{3}\tan\tfrac{\pi}{6} - \sqrt{3}\left(\tfrac{\pi}{6}\right)\right) - \left(\sqrt{3}\tan 0 - \sqrt{3}(0)\right)$$

$$\left(\sqrt{3}\left(\tfrac{\sqrt{3}}{3}\right) - \tfrac{\sqrt{3}\pi}{6}\right) - (0 - 0)$$

$$\boxed{1 - \frac{\sqrt{3}\pi}{6}}$$