Calculus Section 8.5 Partial Fractions (Quadratic Factors)

-Use partial fraction decomposition with quadratic factors

Homework: page 549 #'s 12, 17, 19

When using partial fractions with linear factors, making good choices for values of x can immediately yield values for your coefficients. However, with quadratic factors, a system of linear equations usually has to be solved, regardless of the choice of x.

Example) Distinct Linear and Quadratic Factors

Find $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$ $= \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4}$

$x(x-1)(x^2+4)$

$2x^3-4x-8 = A(x-1)(x^2+4) + B(x)(x^2+4) + (Cx+D)(x)(x-1)$

Let $x=0$

$-8 = A(-1)(4) + B(0)(4) + (0C+D)(0)(-1)$

$-8 = -4A$

$2 = A$

Let $x=1$

$-10 = A(0)(5) + B(1)(5) + (C+D)(1)(0)$

$-10 = 5B$

$-2 = B$

Let $x=-1$

$-6 = A(-2)(5) + B(-1)(5) + (-C+D)(-1)(-2)$

$-6 = -10A - 5B - 2C + 2D$

$-6 = -10(2) - 5(-2) - 2C + 2D$

$-6 = -20 + 10 - 2C + 2D$

$4 = -2C + 2D$

$2 = -C + D$

Let $x=2$

$0 = A(1)(8) + B(2)(8) + (2C+D)(2)(1)$

$0 = 8A + 16B + 4C + 2D$

$0 = 8(2) + 16(-2) + 4C + 2D$

$0 = 16 - 32 + 4C + 2D$

$16 = 4C + 2D$

$8 = 2C + D$

$\begin{cases} -C+D=2 \\ 2C+D=8 \end{cases}$

$2D = 8$

$D = 4$

$C = 2$

$\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx = 2\int \frac{1}{x}dx - 2\int \frac{1}{x-1}dx + \int \frac{2x+4}{x^2+4}dx$

$= 2\ln|x| - 2\ln|x-1| + \int \frac{2x}{x^2+4}dx + 4\int \frac{1}{x^2+4}dx$

$u = x^2+4$, $du = 2x\,dx$ $\qquad$ $u = x$, $a = 2$, $du = dx$

$= \boxed{2\ln\left|\frac{x}{x-1}\right| + \ln|x^2+4| + 4\left(\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right) + C}$

Example) Repeated Quadratic Factors

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

$$8x^3+13x = (Ax+B)(x^2+2) + Cx + D$$

$$8x^3+13x = Ax^3 + 2Ax + Bx^2 + 2B + Cx + D$$

$$8x^3 + 0x^2 + 13x + 0 = Ax^3 + Bx^2 + (2A+C)x + (2B+D)$$

$A = 8$ $\quad B = 0$

$13 = 2A + C$
$13 = 2(8) + C$
$13 = 16 + C$
$-3 = C$

$0 = 2B + D$
$0 = 2(0) + D$
$0 = D$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \int \frac{8x}{x^2+2}\,dx + \int \frac{-3x}{(x^2+2)^2}\,dx$$

$u = x^2+2$
$du = 2x\,dx$ $\quad \frac{1}{2}du = x\,dx$

$$= 4\int \frac{1}{u}\,du - \frac{3}{2}\int u^{-2}\,du$$

$$= \boxed{4\ln|x^2+2| + \frac{3}{2}\left(\frac{1}{x^2+2}\right) + C}$$