Calculus Section 8.8 Improper Integrals: Infinite Discontinuities

-Evaluate an improper integral that has an infinite limit of integration

-Evaluate an improper integral that has an infinite discontinuity

Homework: page 575 #'s 33 – 36, 39

A function that has a vertical asymptote at x = a has an **infinite discontinuity** at x = a. Integrals with infinite discontinuities must be evaluated with limits to circumvent the requirements of the 1st Fund. Thm. of Calculus.

Definition of Improper Integrals with Infinite Discontinuities

1. If f is continuous on the interval [a, b) and has an infinite discontinuity at b, then $\int_a^b f(x)dx = \lim_{c \to b^-} \int_a^c f(x)dx$

2. If f is continuous on the interval (a, b] and has an infinite discontinuity at a, then $\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$

3. If f is continuous on the interval [a, b], except for some c in (a, b) at which f has an infinite discontinuity, then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

In the first two cases, the improper integral **converges** if the limit exists—otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Example) Improper Integral with an Infinite Discontinuity

Evaluate $\int_0^1 \frac{dx}{\sqrt[3]{x}}$

$$\lim_{a \to 0^+} \int_a^1 x^{-1/3}\,dx$$

$$\lim_{a \to 0^+} \left[\frac{3}{2}x^{2/3}\right]_a^1$$

$$\lim_{a \to 0^+} \left[\frac{3}{2}(1)^{2/3} - \frac{3}{2}(a)^{2/3}\right]$$

$$\frac{3}{2} - \frac{3}{2}(0^+)^{2/3}$$

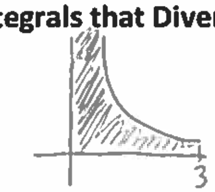
$$\frac{3}{2} - 0$$

$$\frac{3}{2}$$

The integral converges to $3/2$

Example) Improper Integrals that Diverge

Evaluate $\int_0^3 \frac{dx}{x^3}$



$$\lim_{a \to 0^+} \int_a^3 x^{-3}\,dx$$

$$\lim_{a \to 0^+} \left[-\frac{1}{2}x^{-2}\right]_a^3$$

$$\lim_{a \to 0^+} \left[\frac{-1}{2(3)^2} - \frac{-1}{2a^2}\right]$$

$$\frac{-1}{18} + \frac{1}{2(0^+)^2}$$

$$\frac{-1}{18} + \frac{1}{0^+}$$

$$\frac{-1}{18} + \infty$$

$$\infty$$

The integral diverges

Example) Improper Integral with Interior Discontinuity

Evaluate $\int_{-1}^{2} \frac{dx}{x^3}$

$$\lim_{a\to 0^-} \int_{-1}^{a} x^{-3}dx + \lim_{b\to 0^+} \int_{b}^{2} x^{-3}dx$$

$$\lim_{a\to 0^-} \left[-\frac{1}{2}x^{-2}\right]_{-1}^{a} + \lim_{b\to 0^+} \left[-\frac{1}{2}x^{-2}\right]_{b}^{2}$$

$$\lim_{a\to 0^-} \left[\frac{-1}{2a^2} - \frac{-1}{2(-1)^2}\right] + \lim_{b\to 0^+} \left[\frac{-1}{2(2)^2} - \frac{-1}{2(b)^2}\right]$$

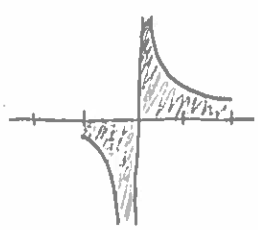
$$\frac{-1}{2(0^-)^2} + \frac{1}{2} - \frac{1}{8} + \frac{1}{2(0^+)^2}$$

$$\frac{-1}{0^+} + \frac{1}{2} - \frac{1}{8} + \frac{1}{0^+}$$

$$-\infty + \frac{1}{2} - \frac{1}{8} + \infty$$

Diverges

The integral diverges because one of the limits was infinite



Example) Doubly Improper Integral

Evaluate $\int_{0}^{\infty} \frac{dx}{\sqrt{x}(x+1)}$

$$\int_{0}^{1} \frac{1}{\sqrt{x}(x+1)}dx + \int_{1}^{\infty} \frac{1}{\sqrt{x}(x+1)}dx \quad \leftarrow 1 \text{ is arbitrary}$$

$$\lim_{a\to 0^+} \int_{a}^{1} \frac{1}{\sqrt{x}(x+1)}dx + \lim_{b\to\infty} \int_{1}^{b} \frac{1}{\sqrt{x}(x+1)}dx$$

$$x+1 = (\sqrt{x})^2 + (1)^2$$

$$u = \sqrt{x}$$

$$du = \frac{1}{2}x^{-1/2}dx$$

$$2du = \frac{1}{\sqrt{x}}dx$$

$$2\lim_{a\to 0^+} \int_{a}^{1} \frac{1}{u^2+1^2}du + 2\lim_{b\to\infty} \int_{1}^{b} \frac{1}{u^2+1^2}du$$

$$2\lim_{a\to 0^+} \left[\arctan(\sqrt{x})\right]_{a}^{1} + 2\lim_{b\to\infty} \left[\arctan\sqrt{x}\right]_{1}^{b}$$

$$2\lim_{a\to 0^+} \left[\arctan(1) - \arctan\sqrt{a}\right] + 2\lim_{b\to\infty} \left[\arctan\sqrt{b} - \arctan(1)\right]$$

$$\cancel{2\arctan(1)} - 2\arctan\sqrt{0^+} + 2\arctan\sqrt{\infty} - \cancel{2\arctan(1)}$$

$$-2\arctan(0^+) + 2\arctan(\infty)$$

$$0 + 2\left(\frac{\pi}{2}\right)$$

$$\pi$$

The integral converges to π