8.8 Improper Integrals (Infinite Limits)

Pg. 575 #'s 17-27 odd, 31, 71

17) $\int_1^\infty \frac{1}{x^3}dx$

$\lim_{a\to\infty} \int_1^a x^{-3}dx$

$\lim_{a\to\infty} -\frac{1}{2}x^{-2}\Big|_1^a$

$\lim_{a\to\infty} \left[\frac{-1}{2a^2} - \frac{-1}{2}\right]$

$-\frac{1}{\infty} + \frac{1}{2} = \boxed{\frac{1}{2}}$ $\boxed{\text{converges}}$

19) $\int_1^\infty \frac{3}{\sqrt[3]{x}}dx$

$\lim_{a\to\infty} \int_1^a 3x^{-1/3}dx$

$\lim_{a\to\infty} \frac{9}{2}x^{2/3}\Big|_1^a$

$\lim_{a\to\infty} \left[\frac{9}{2}(a)^{2/3} - \frac{9}{2}\right]$

$\infty - 4.5 = \boxed{\infty}$ $\boxed{\text{diverges}}$

21) $\int_{-\infty}^0 xe^{-4x}dx$

$\lim_{a\to-\infty} \int_a^0 xe^{-4x}dx$

signs	u	dv
+	x	e^{-4x}
−	1	$-\frac{1}{4}e^{-4x}$
+	0	$\frac{1}{16}e^{-4x}$

$\lim_{a\to-\infty} \left[-\frac{1}{4}xe^{-4x} - \frac{1}{16}e^{-4x}\right]_a^0$

$\lim_{a\to-\infty} \left[(0-\frac{1}{16}) - (-\frac{1}{4}ae^{-4a} - \frac{1}{16}e^{-4a})\right]$

$-\frac{1}{16} + \frac{1}{4}(\infty)(\infty) + \frac{1}{16}(\infty)$

$\boxed{\infty}$ $\boxed{\text{diverges}}$

23) $\int_0^\infty x^2e^{-x}dx$

$\lim_{a\to\infty} \int_0^a x^2e^{-x}dx$

signs	u	dv
+	x^2	e^{-x}
−	$2x$	$-e^{-x}$
+	2	e^{-x}
−	0	$-e^{-x}$

$\lim_{a\to\infty} \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^a$

$\lim_{a\to\infty} \left[(-a^2e^{-a} - 2ae^{-a} - 2e^{-a}) - (0-0-2)\right]$

$\lim_{a\to\infty} \left[\frac{-a^2}{e^a} - \frac{2a}{e^a} - \frac{2}{e^a}\right] + 2 = \frac{\infty}{\infty} + 2$

l'Hop

$\lim_{a\to\infty} \left[\frac{-2a}{e^a} - \frac{2}{e^a} - \frac{0}{e^a}\right] = \frac{\infty}{\infty} + 2 \xrightarrow{\text{l'Hop}} \lim_{a\to\infty} \left[\frac{-2}{e^a} - \frac{0}{e^a}\right] = \frac{-2}{\infty} + 2 = 0 + 2 = \boxed{2}$ $\boxed{\text{converges}}$

25) $\int_4^\infty \frac{1}{x(\ln x)^3}dx$

$\lim_{a\to\infty} \int_4^a \frac{1}{x(\ln x)^3}dx$ $\quad u = \ln x \quad u(a) = \ln a$

$du = \frac{1}{x}dx \quad u(4) = \ln 4$

$\lim_{a\to\infty} \int_{\ln 4}^{\ln a} u^{-3}du$

$\lim_{a\to\infty} \left[-\frac{1}{2}u^{-2}\right]_{\ln 4}^{\ln a} \longrightarrow \lim_{a\to\infty} \left[\frac{-1}{2(\ln a)^2} - \frac{-1}{2(\ln 4)^2}\right]$

$\frac{-1}{2(\infty)} + \frac{1}{2(\ln 4)^2}$

$\boxed{\frac{1}{2(\ln 4)^2}}$ $\boxed{\text{converges}}$

27) $\int_{-\infty}^{\infty} \frac{4}{16+x^2} dx$

$\lim_{a \to -\infty} 4\int_a^0 \frac{1}{16+x^2} dx + \lim_{b \to \infty} 4\int_0^b \frac{1}{16+x^2} dx$

$u = x \quad du = dx \quad a = 4$

$\lim_{a \to -\infty} 4\left[\frac{1}{4}\arctan\frac{x}{4}\right]_a^0 + \lim_{b \to \infty} 4\left[\frac{1}{4}\arctan\frac{x}{4}\right]_0^b$

$\lim_{a \to -\infty} [\arctan(0) - \arctan(a)] + \lim_{b \to \infty} [\arctan(b) - \arctan(0)]$

$\left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right)$

$\boxed{\pi}$

$\boxed{\text{converges}}$

31) $\int_0^{\infty} \cos\pi x \, dx$

$\lim_{a \to \infty} \int_0^a \cos\pi x \, dx$ $\quad u = \pi x$, $du = \pi dx$, $\frac{1}{\pi} du = dx$

$\lim_{a \to \infty} \left[\frac{1}{\pi}\sin\pi x\right]_0^a$

$\lim_{a \to \infty} \left[\frac{1}{\pi}\sin\pi a - \frac{1}{\pi}\sin(0)\right]$

$\frac{1}{\pi}\sin(\infty) - 0$

some value between $\frac{-1}{\pi}$ and $\frac{1}{\pi}$

$\boxed{\text{diverges}}$

71) a) $\int_0^{\infty} e^{-x} dx$

$\lim_{a \to \infty} \int_0^a e^{-x} dx$

$\lim_{a \to \infty} -e^{-x}\Big|_0^a = \lim_{a \to \infty} \left[-e^{-a} - -e^0\right]$

$-e^{-\infty} + 1 = \boxed{1}$

b) $\lim_{a \to \infty} \pi\int_0^a (e^{-x})^2 dx = \lim_{a \to \infty} \pi\int_0^a e^{-2x} dx$

$\lim_{a \to \infty} \left[-\frac{\pi}{2} e^{-2x}\right]_0^a = \lim_{a \to \infty} \left[-\frac{\pi}{2}e^{-2a} - -\frac{\pi}{2}e^0\right]$

$-\frac{\pi}{2}e^{-\infty} + \frac{\pi}{2} = \boxed{\frac{\pi}{2}}$

c) $\lim_{a \to \infty} 2\pi\int_0^a x(e^{-x}) dx$

signs	u	dv
+	x	e^{-x}
−	1	$-e^{-x}$
+	0	e^{-x}

$2\pi \lim_{a \to \infty} \left[-xe^{-x} - e^{-x}\right]_0^a$

$2\pi \lim_{a \to \infty} \left[\left(\frac{-a}{e^a} - \frac{1}{e^a}\right) - (0 - 1)\right] = \left[\frac{\infty}{\infty} + 1\right]2\pi$

l'Hop

$2\pi\left(\lim_{a \to \infty}\left[\frac{-1}{e^a} - \frac{1}{e^a}\right] + 1\right) = 2\pi(0+1) = \boxed{2\pi}$