Calculus Section 8.8 Improper Integrals (Infinite Limits)

-Evaluate an improper integral that has an infinite limit of integration
-Evaluate an improper integral that has an infinite discontinuity

Homework: page 575 #'s 17 – 27 odd, 31, 71

The definition of a definite integral $\int_a^b f(x)dx$ requires that the interval [a, b] be finite. Furthermore, the Fundamental Theorem of Calculus requires that *f* be continuous on [a, b]. We use **improper integrals** to get around both problems by using limits to artificially set the limits of integration to be definite.

Definition of Improper Integrals with Infinite Integration Limits

1) If *f* is continuous on the interval [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$

2) If *f* is continuous on the interval (-∞, b], then $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty}\int_a^b f(x)dx$

3) If *f* is continuous on the interval (-∞,∞), then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$ where c is any real number.

An integral is said to **converge** if the integral equates to a finite value. An integral **diverges** if it equals infinity or cannot be determined.

Example) An Improper Integral that Diverges

Evaluate $\int_1^\infty \frac{dx}{x}$

$\lim_{a\to\infty}\int_1^a \frac{1}{x}dx$

$\lim_{a\to\infty}\left[\ln|x|\right]_1^a$

$\lim_{a\to\infty}\left[\ln|a| - \ln|1|\right]$

$\ln|\infty| - 0$

$\boxed{\infty}$

$\boxed{\text{diverges}}$

Example) An Improper Integral that Converges

Evaluate $\int_{-\infty}^0 e^x dx$

$\lim_{a\to-\infty}\int_a^0 e^x dx$

$\lim_{a\to-\infty}\left[e^x\right]_a^0$

$\lim_{a\to-\infty}\left[e^0 - e^a\right]$

$e^0 - e^{-\infty}$

$1 - 0$

$\boxed{1}$

$\boxed{\text{converges}}$

Example)

Evaluate $\int_0^\infty \frac{1}{x^2+1}dx$

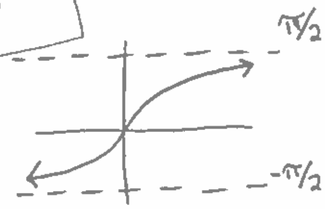
$$\lim_{a\to\infty} \int_0^a \frac{1}{x^2+1}dx \qquad \begin{matrix} u=x \\ du=dx \end{matrix} \qquad a=1$$

$$\lim_{a\to\infty} \Big[\arctan x\Big]_0^a$$

$$\lim_{a\to\infty} \Big[\arctan(a) - \arctan(0)\Big]$$

$$\frac{\pi}{2} - 0$$

$$\boxed{\frac{\pi}{2} \text{ converges}}$$



Example) Using L'Hôpital's Rule

Evaluate $\int_1^\infty (1-x)e^{-x}dx$

Signs	u	dv
+	$1-x$	e^{-x}
−	-1	$-e^{-x}$
+	0	e^{-x}

$$\lim_{a\to\infty}\Big[-(1-x)e^{-x} + e^{-x}\Big]_1^a$$

$$\lim_{a\to\infty}\Big[\frac{x-1}{e^x} + \frac{1}{e^x}\Big]_1^a$$

$$\lim_{a\to\infty}\Big[\Big(\frac{a-1}{e^a} + \frac{1}{e^a}\Big) - \Big(\frac{0}{1} + \frac{1}{e}\Big)\Big]$$

l'Hop

$$\frac{\infty}{\infty} + \frac{1}{\infty} - 0 - \frac{1}{e} = \frac{\infty}{\infty} - \frac{1}{e}$$

$$\lim_{a\to\infty}\Big[\frac{1}{e^a} + \frac{1}{e^a}\Big] - \frac{1}{e} = \frac{1}{\infty} + \frac{1}{\infty} - \frac{1}{e}$$

$$= \boxed{-\frac{1}{e}}$$

$$\boxed{\text{converges}}$$

Example) Infinite Upper and Lower Limits of Integration

Evaluate $\int_{-\infty}^{\infty} \frac{e^x}{1+e^{2x}}dx$

$$\lim_{a\to-\infty}\int_a^0 \frac{e^x}{1+e^{2x}}dx + \lim_{b\to\infty}\int_0^b \frac{e^x}{1+e^{2x}}dx$$

$$\begin{matrix} u=e^x \\ du=e^x dx \end{matrix} \qquad a=1$$

$$\lim_{a\to-\infty}\Big[\arctan e^x\Big]_a^0 + \lim_{b\to\infty}\Big[\arctan e^x\Big]_0^b$$

$$\lim_{a\to-\infty}\Big[\arctan(e^0) - \arctan(e^a)\Big] + \lim_{b\to\infty}\Big[\arctan(e^b) - \arctan(e^0)\Big]$$

$$\cancel{\arctan(1)} - \arctan(0) + \arctan(\infty) - \cancel{\arctan(1)}$$

$$0 + \frac{\pi}{2}$$

$$\boxed{\frac{\pi}{2} \text{ converges}}$$