Calculus Section 9.10 Taylor and Maclaurin Series

-Understand the definition of a power series

-Create Taylor and Maclaurin series

Homework: page 673 #'s 1 – 7

All functions, including some important functions like f(x) = e^x, can be approximated using a polynomial. The polynomial itself will not be exactly the same value of the function, but can at times be so close that there really isn't much difference in the values of f(x) at certain values of x. A power series approximation get better as more terms are added to it.

Function	x = 0	x = 1	x = 2
$f(x) = e^x$	1	2.7183	7.3891
$P(x) = 1 + x$	1	2	3
$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$	1	2.666	6.333
$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$	1	2.7167	7.2667

The **degree** of a power series is defined by the power on the highest exponent. For example,

$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ has degree 3.

$P(x) = 1 - \frac{x^5}{2}$ has degree 5.

$P(x) = 1 + x$ has degree 1.

$P(x) = 5$ has degree zero.

$P(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots + \frac{x^n}{n!}$ has degree n.

Definition of Power Series

If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

is called a **power series**. More generally, an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

is called a **power series centered at c**, where c is a constant.

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$ is centered at $x = 0$.

$\sum_{n=0}^{\infty} (-1)^n (x+1)^n = 1 - (x+1) + (x+1)^2 - (x+1)^3 \ldots$ is centered at $x = -1$.

$\sum_{n=0}^{\infty} \frac{1}{n}(x-1)^n = (x-1) + \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \ldots$ is centered at $x = 1$.

Any convergent power series for a function can be written from the same basic form. That form is called the Taylor series; named after the mathematician Brook Taylor.

Definitions of Taylor and Maclaurin Series

If a function *f* has derivatives of all orders at x = c, then the series

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n = f(c)+f'(c)(x-c)+\frac{f''(c)(x-c)^2}{2!}+\frac{f'''(c)(x-c)^3}{3!}+\ldots+\frac{f^{(n)}(c)(x-c)^n}{n!}+\ldots$$

is called the **Taylor series for f(x) at c.** If the center of the function is at zero, c = 0, then the series is called a **Maclaurin series for *f*.** It is a special form of the Taylor series.

Taylor and Maclaurin series have an infinite number of terms. A **Taylor** or **Maclaurin polynomial** can have a finite number of terms.

Examples)

1) Form the 1st, 2nd, and 3rd degree Maclaurin polynomial, as well as the general term for $f(x) = e^x$.

$f(x)=e^x$ $\quad f(0)=1$ $\quad P_1(x) = 1 + x$

$f'(x)=e^x$ $\quad f'(0)=1$ $\quad P_2(x) = 1 + x + \frac{x^2}{2!}$

$f''(x)=e^x$ $\quad f''(0)=1$ $\quad P_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$f'''(x)=e^x$ $\quad f'''(0)=1$

General term: $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$0! = 1$

$1! = 1$

$2! = 2$

$3! = 6$

2) Form a 4th degree Taylor polynomial for $f(x) = e^x$ centered at x = 1. Also, determine the general term.

$f(x)=e^x$ $\quad f(1)=e$ $\quad P_4(x) = e + e(x-1) + \frac{e(x-1)^2}{2!} + \frac{e(x-1)^3}{3!} + \frac{e(x-1)^4}{4!}$

$f'(x)=e^x$ $\quad f'(1)=e$ $\quad \sum_{n=0}^{\infty}\frac{e(x-1)^n}{n!}$

$f''(x)=e^x$ $\quad f''(1)=e$

$f'''(x)=e^x$ $\quad f'''(1)=e$

$f''''(x)=e^x$ $\quad f''''(1)=e$

3) Form the Maclaurin series for f(x) = sin(x).

$f(x) = \sin x$ $\quad f(0) = 0$

$f'(x) = \cos x$ $\quad f'(0) = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -1$

$f^{(4)}(0) = 0$

$f^{(5)}(0) = 1$

$f^{(6)}(0) = 0$

$f^{(7)}(0) = -1$

$$P(x) = 0 + x + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!} + \frac{0x^6}{6!} - \frac{x^7}{7!} + \dots$$

$$P(x) = \underset{n=0}{x} - \underset{n=1}{\frac{x^3}{3!}} + \underset{n=2}{\frac{x^5}{5!}} - \underset{n=3}{\frac{x^7}{7!}} + \dots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

4) Form the Maclaurin series for f(x) = cos(x).

$f(x) = \cos x$ $\quad f(0) = 1$

$f'(x) = -\sin x$ $\quad f'(0) = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -1$

$f'''(x) = \sin x$ $\quad f'''(0) = 0$

$$P(x) = 1 + 0x - \frac{x^2}{2!} + \frac{0x^3}{3!} + \frac{x^4}{4!} + \frac{0x^5}{5!} - \frac{x^6}{6!} + \dots$$

$$P(x) = \underset{n=0}{1} - \underset{n=1}{\frac{x^2}{2!}} + \underset{n=2}{\frac{x^4}{4!}} - \underset{n=3}{\frac{x^6}{6!}} + \dots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

You must <u>memorize</u> the Maclaurin series for e^x, sin x, and cos x.