Calculus Section 9.10 Taylor/Maclaurin Manipulation

-Use properties of series and power series to transform a series
-Use properties of series and power series to solve complex problems

Homework: page 673 #'s 27 – 35 odd, 47, 59, 65

We can manipulate Taylor (and Maclaurin) series in order to find other series by using the following techniques:

1) Substitute into the series
2) Multiply or divide the series by a constant or variable
3) Add or subtract two series
4) Differentiate or integrate a series

Examples)

Find the first four non-zero terms of the Maclaurin series f(x) = sinxcosx.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ $\qquad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$x\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) = x - \frac{x^3}{2} + \frac{x^5}{24} - \frac{x^7}{720}$

$-\frac{x^3}{6}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) = -\frac{x^3}{6} + \frac{x^5}{12} - \frac{x^7}{144}$

$\frac{x^5}{120}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) = \frac{x^5}{120} - \frac{x^7}{240}$

$\frac{x^7}{5040}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right) = \frac{x^7}{5040}$

$\sin x \cos x = x - \frac{2x^3}{3} + \frac{2x^5}{15} - \frac{31x^7}{2520}$

Find a Maclaurin series and the first four non-zero terms of f(x) = $\sin^2 x$.

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!}$

$\frac{1-\cos 2x}{2} = \frac{1 - \left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!}\right)}{2}$

$\frac{1-\cos 2x}{2} = \frac{2x^2}{2!} - \frac{2^3x^4}{4!} + \frac{2^5x^6}{6!} - \frac{2^7x^8}{8!}$

$\sin^2 x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n}}{(2n)!}$

Find a Maclaurin series and the first four non-zero terms of f(x) = e^{2x}.

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{2x} = 1 + (2x) + \frac{(2x)^2}{2} + \frac{(2x)^3}{3!} \qquad \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$

Find a Maclaurin series and the first four non-zero terms of $f(x) = \dfrac{e^x + e^{-x}}{2}$.

$\dfrac{e^x + e^{-x}}{2} = \cosh(x)$

"hyperbolic cosine"

models a line hanging between two fixed points

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!}$$

$$e^x + e^{-x} = 2 + x^2 + \frac{2x^4}{4!} + \frac{2x^6}{6!}$$

$$\frac{e^x + e^{-x}}{2} = \frac{2 + x^2 + \frac{2x^4}{4!} + \frac{2x^6}{6!}}{2} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} \qquad \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Find the first 6 non-zero terms of the Maclaurin series for $f(x) = e^x + \cos x$. Approximate the value of $f'(0.5)$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!}$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$e^x + \cos x = 2 + x + \frac{x^3}{3!} + \frac{2x^4}{4!} + \frac{x^5}{5!} + \frac{x^7}{7!}$$

$$f'(x) = 1 + \frac{3x^2}{6} + \frac{8x^3}{24} + \frac{5x^4}{120} + \frac{7x^6}{5040}$$

$$f'(0.5) = 1 + \frac{3(.5)^2}{6} + \frac{8(.5)^3}{24} + \frac{5(.5)^4}{120} + \frac{7(.5)^6}{5040} \approx 1.169$$

Use a 6^{th} degree Maclaurin polynomial to approximate the value of $\int_0^1 \sin(x^2)dx$. What is the maximum error of this approximation?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!}$$

$$\int_0^1 \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}\right)dx$$

$$\left[\boxed{\frac{1}{3}x^3 - \frac{1}{42}x^7} + \frac{x^{11}}{1320}\right]_0^1$$

$$\left(\frac{1}{3} - \frac{1}{42}\right) - (0 - 0)$$

$$\int_0^1 \sin(x^2)dx \approx .309523$$

The maximum error is $\frac{1}{1320}$