9.10 Taylor/Mac Day 2

1) 5th degree Mac $f(x) = \sin(3x)$

$f(x) = \sin(3x)$ $\quad f(0) = 0$

$f'(x) = 3\cos(3x)$ $\quad f'(0) = 3$

$f''(x) = -9\sin(3x)$ $\quad f''(0) = 0$

$f'''(x) = -27\cos(3x)$ $\quad f'''(0) = -27$

$f^{(4)}(x) = 81\sin(3x)$ $\quad f^{(4)}(0) = 0$

$f^{(5)}(x) = 243\cos(3x)$ $\quad f^{(5)}(0) = 243$

$$\boxed{P(x) = 3x - \frac{27x^3}{3!} + \frac{243x^5}{5!}}$$

2) 5th degree Taylor at $x=2$ $\quad f(x) = \ln(x-1)$

$f(x) = \ln(x-1)$ $\quad f(2) = \ln(1) = 0$

$f'(x) = \frac{1}{x-1}$ $\quad f'(2) = \frac{1}{1} = 1$

$f''(x) = \frac{-1}{(x-1)^2}$ $\quad f''(2) = \frac{-1}{1^2} = -1$

$f'''(x) = \frac{2(x-1)}{(x-1)^4} = \frac{2}{(x-1)^3}$ $\quad f'''(2) = \frac{2}{1} = 2$

$f^{(4)}(x) = \frac{-2(3(x-1)^2)}{(x-1)^6} = \frac{-6}{(x-1)^4}$ $\quad f^{(4)}(2) = \frac{-6}{1} = -6$

$f^{(5)}(x) = \frac{24}{(x-1)^5}$ $\quad f^{(5)}(2) = \frac{24}{1} = 24$

$$\boxed{P(x) = (x-2) - \frac{(x-2)^2}{2!} + \frac{2(x-2)^3}{3!} - \frac{6(x-2)^4}{4!} + \frac{24(x-2)^5}{5!}}$$

3) 4th degree Taylor at $x=4$ $\quad f(x) = e^{(x-4)}$

$f(x) = e^{x-4}$ $\quad f(4) = 1$

$f'(x) = e^{x-4}$ $\quad f'(4) = 1$

$f''(x) = e^{x-4}$ $\quad f''(4) = 1$

$f'''(x) = e^{x-4}$ $\quad f'''(4) = 1$

$f^{(4)}(x) = e^{x-4}$ $\quad f^{(4)}(4) = 1$

$$\boxed{P(x) = 1 + (x-4) + \frac{(x-4)^2}{2!} + \frac{(x-4)^3}{3!} + \frac{(x-4)^4}{4!}}$$

4) a) $P_3(x) = -3 + 7x^2 - 2x^3$

$\boxed{f(0) = -3}$ $\quad \boxed{f'(0) = 0}$

$\frac{f''(0)}{2!} = 7$ $\quad \frac{f'''(0)}{6} = -2$

$\boxed{f''(0) = 14}$ $\quad \boxed{f'''(0) = -12}$

b) By the 2nd derivative test, $f(x)$ has a minimum at $x=0$ b/c the graph is concave up

5) a) 4th deg. Mac $\quad f(x) = \cos x$

$$\boxed{P(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}}$$

b) $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}\right)}{x^2}$

$\lim_{x\to 0} \frac{\frac{x^2}{2} - \frac{x^4}{4!}}{x^2}$

$\lim_{x\to 0} \frac{1}{2} - \frac{x^2}{4!} = \boxed{\frac{1}{2}}$

6) a) 3rd degree Mac $\quad f(x) = \frac{1}{1-2x}$

$f(x) = \frac{1}{1-2x}$ $\quad f(0) = 1$

$f'(x) = \frac{-(-2)}{(1-2x)^2} = \frac{2}{(1-2x)^2}$ $\quad f'(0) = 2$

$f''(x) = \frac{-2(2(1-2x))(2)}{(1-2x)^4} = \frac{-8}{(1-2x)^3}$ $\quad f''(0) = -8$

$f'''(x) = \frac{-(-8)(3(1-2x)^2)(2)}{(1-2x)^6} = \frac{48}{(1-2x)^4}$ $\quad f'''(0) = 48$

$$\boxed{P(x) = 1 + 2x - \frac{8x^2}{2!} + \frac{48x^3}{3!}}$$

6) b) $\lim\limits_{x\to 0} \dfrac{f(x)-1}{x} = \dfrac{\left(1+2x-\frac{8x^2}{2}+\frac{48x^3}{3!}\right)-1}{x}$

$$\lim_{x\to 0} \frac{2x - \frac{8x^2}{2} + \frac{48x^3}{3!}}{x}$$

$$\lim_{x\to 0} 2 - \frac{8x}{2} + \frac{48x^2}{3!} = \boxed{2}$$

7) a) 7^{th} deg. Mac $f(x) = \sin x$

$$\boxed{P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$$

b) $\displaystyle\int_0^1 \frac{\sin t}{t}\,dt = \int_0^1 \frac{\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!}\right)}{t}\,dt$

$$\int_0^1 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!}\,dt$$

$$t - \frac{t^3}{3\cdot 3!} + \frac{t^5}{5\cdot 5!} - \frac{t^7}{7\cdot 7!}\Big|_0^1$$

$$\left(\left[1 - \frac{1}{18}\right] + \frac{1}{600} - \frac{1}{35280}\right) - (0)$$

$\frac{17}{18}$ is an approx. with less than $\frac{1}{500}$ error b/c our first neglected term is $\frac{1}{600} < \frac{1}{500}$ and the terms are decreasing.

c) $\displaystyle\int_0^1 \frac{\sin t}{t}\,dt = .946083$

$$\frac{17}{18} = .94444$$

error $= .946083 - .94444$

$\boxed{\text{error} = .00164}$