Taylor Polynomials and Approximations, Day 2

Yesterday we learned:

Definition of an nth-degree Taylor polynomial:

If f has n derivatives at $x=c$, then the polynomial

$$P_n(x)=f(c)+f'(c)(x-c)+\frac{f''(c)}{2!}(x-c)^2+\ldots+\frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the nth-degree Taylor polynomial for f at c, named after Brook Taylor, an English mathematician.

If $c=0$, then $P_n(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n$ is called the nth-degree Maclaurin polynomial for f, named after another English mathematician, Colin Maclaurin.

Ex. (a) Find the third-degree Maclaurin polynomial for $f(x)=e^x$.

$P_3(x)=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}$

(b) Use your answer to (a) to find: $\lim_{x\to 0}\frac{f(x)-1}{2x}$

$\lim_{x\to 0}\frac{(\cancel{1}+x+\frac{x^2}{2!}+\frac{x^3}{3!})\cancel{-1}}{2x}=\lim_{x\to 0}=\frac{1+\frac{x}{2!}+\frac{x^2}{3!}}{2}=\frac{1}{2}$

Ex. Suppose that the function $f(x)$ is approximated near $x=0$ by a third-degree Taylor polynomial $P_3(x)=2-5x^2+8x^3$.

(a) Find the value of $f(0)$, $f'(0)$, $f''(0)$, and $f'''(0)$.

$f(0)=2 \quad f'(0)=0 \quad \frac{f''(0)x^2}{2!}=-5x^2 \quad \frac{f'''(0)x^3}{3!}=8x^3$

$\frac{f''(0)}{2!}=-5 \quad \frac{f'''(0)}{3!}=8$

$f''(0)=-5\cdot 2!=-10 \quad f'''(0)=8\cdot 3!=48$

☆☆☆ (b) Does f have a local maximum, a local minimum, or neither at $x=0$? Justify your answer.

Not enough info for 1st deriv test / $x'=0$ but switch from + to − or − to +?

2nd Derivative Test: ① $f'(a)=0$

② $f''(a)=+$, concave up, relative minimum at $(a,f(a))$

③ $f''(a)=-$, concave down, relative max at $(a,f(a))$

④ $f''(a)=0$, inconclusive

$f'(0)=0$ and $f''(0)=-10$

∴ f has a local max at $x=0$

Ex. (a) Find the sixth-degree Maclaurin polynomial for $f(x) = \cos x$.

$f(x) = \cos(x)$ $\quad f(0) = 1$

$f'(x) = -\sin(x)$ $\quad f'(0) = 0$

$f''(x) = -\cos(x)$ $\quad f''(0) = -1$

$f'''(x) = \sin(x)$ $\quad f'''(0) = 0$

$f^{(4)}(x) = \cos(x)$ $\quad f^{(4)}(0) = 1$

$$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

(b) Use your answer to (a) to approximate the value of $\int_0^1 \frac{1-\cos t}{t^2}\,dt$ so that the error in your approximation is less than $\frac{1}{500}$. Justify your answer.

← too hard to integrate, use P instead to make easier

$$\int_0^1 \frac{1-\left(1-\frac{t^2}{2}+\frac{t^4}{4!}-\frac{t^6}{6!}\right)}{t^2}\,dt = \int_0^1 \frac{\frac{t^2}{2}-\frac{t^4}{4!}+\frac{t^6}{6!}}{t^2}\,dt = \int_0^1 \frac{1}{2} - \frac{t^2}{4!} + \frac{t^4}{6!}\,dt$$

$$= \left[\frac{1}{2}t - \frac{t^3}{3 \cdot 4!} + \frac{t^5}{5 \cdot 6!}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{72} + \frac{1}{5(720)}$$

Since terms are ① alt, ② decr. in mag., ③ have lim of zero, $|\text{Error}| < \frac{1}{5(720)} < \frac{1}{500}$ if I estimate the answer to be $\frac{1}{2} - \frac{1}{72} = \frac{35}{72}$

(c) Use your calculator to find the actual value of $\int_0^1 \frac{1-\cos t}{t^2}\,dt$. What is the error in the approximation you found in (b)?

$\frac{35}{72} = .486\ldots$

$\int_0^1 \frac{1-\cos t}{t^2}\,dt = .486\ldots$

$|\text{Error}| = .000274$