Calculus Section 9.1 Sequences

-List the terms of a sequence and write a sequence.
-Determine whether a sequence converges or diverges

Homework: page 592 #'s 1 – 7 odd, 18, 19, 29 – 49 odd

Mathematically, a **sequence** is defined as a function whose domain is the set of positive integers. Each integer is mapped to a term of the sequence.

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & \dots & n, & \dots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_1 & a_2 & a_3 & a_4 & a_5 & \dots & a_n & \dots \end{array}$$

The numbers $a_1, a_2, a_3, \dots, a_n, \dots$ are the **terms** of the sequence. The number a_n is called the **nth term** of the sequence, and the entire sequence is notated using curly-brackets: $\{a_n\}$.

recursive: new terms defined by previous terms

Example) List the terms of each sequence:

1) $\{a_n\} = \{3 + (-1)^n\}$

$n=1 \quad n=2 \quad n=3 \quad n=4$

$2, \quad 4, \quad 2, \quad 4, \dots$

2) $\{a_n\} = \left\{\frac{1}{2^n}\right\}$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

3) $\{d_n\}$ is $d_{n+1} = d_n - 5$, $d_1 = 25$

$25, 20, 15, 10, \dots$

A primary focus of this chapter concerns sequences whose terms approach limiting values. These sequences are said to **converge**. For instance, the sequence $\left\{\frac{1}{2^n}\right\}$ converges to 0.

Evaluate $\lim_{n\to\infty}\{a_n\}$ to determine whether (and to what) a sequence converges.

Example) Determine whether each sequence converges

1) $\{a_n\} = \{3 + (-1)^n\}$

$\lim_{n\to\infty} (3 + (-1)^n) = ?\ 2 \text{ or } 4$

the sequence diverges

2) $\{b_n\} = \left\{\frac{n}{1-2n}\right\}$

L'Hop

$\lim_{n\to\infty} \frac{n}{1-2n} = \frac{\infty}{\infty}$

$\lim_{n\to\infty} \frac{1}{-2} = -\frac{1}{2}$

the sequence converges to $-1/2$

3) $\{c_n\} = \frac{n^2}{2^n - 1}$

$\lim_{n\to\infty} \frac{n^2}{2^n-1} = \frac{\infty}{\infty}$

$\lim_{n\to\infty} \frac{2n}{\ln(2)2^n} = \frac{\infty}{\infty}$

$\lim_{n\to\infty} \frac{2}{[\ln(2)]^2 2^n} = \frac{2}{\infty} = 0$

the sequence converges to zero

4) $\{a_n\} = \left\{\frac{\ln(n)}{n}\right\}$

$\lim_{n\to\infty} \frac{\ln(n)}{n} = \frac{\infty}{\infty}$

$\lim_{n\to\infty} \frac{1/n}{1} = \frac{1}{n} = \frac{1}{\infty} = 0$

the sequence converges to zero

5) $\{b_n\} = \frac{(n+1)!}{n!}$

$\lim_{n\to\infty} \frac{(n+1)!}{n!}$

$\lim_{n\to\infty} \frac{(n+1)n!}{n!}$

$\lim_{n\to\infty} n+1 = \infty$

the sequence diverges

6) $\{c_n\} = \frac{(n+1)!}{(n+3)!}$

$\lim_{n\to\infty} \frac{(n+1)!}{(n+3)!} = \frac{(n+1)!}{(n+3)(n+2)(n+1)!}$

$\lim_{n\to\infty} \frac{1}{(n+3)(n+2)} = \frac{1}{\infty} = 0$

The sequence converges to zero

Properties of Limits of Sequences

Let $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$.

1) $\lim_{n\to\infty}(a_n \pm b_n) = L \pm K$

2) $\lim_{n\to\infty}(ca_n) = cL$, c is any real number

3) $\lim_{n\to\infty}(a_n b_n) = LK$

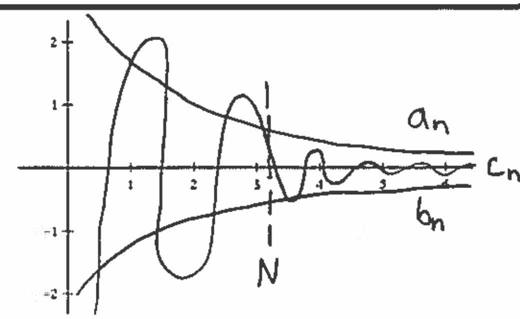
4) $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \frac{L}{K}$, $b_n \neq 0$ and $K \neq 0$

Squeeze Theorem for Sequences

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ and there exists an integer N such that $a_n \le c_n \le b_n$ for all $n > N$, then $\lim_{n\to\infty} c_n = L$.

c_n is trapped in between a_n and b_n.

Since both limits approach zero, c_n does too.



Example) Show that the sequence $\{c_n\} = \left\{(-1)^n \frac{1}{n!}\right\}$ converges, and find its limit.

$$\frac{-1}{n!} \le (-1)^n \frac{1}{n!} \le \frac{1}{n!}$$

$$\lim_{n\to\infty} \frac{-1}{n!} = \frac{1}{\infty} = 0 \qquad \lim_{n\to\infty} \frac{1}{n!} = \frac{1}{\infty} = 0$$

By the Squeeze Theorem $\{c_n\}$ converges to zero

Find the nth Term of a Sequence

Find a sequence $\{a_n\}$ whose first five terms are $\frac{-2}{1}, \frac{8}{2}, \frac{-26}{6}, \frac{80}{24}, \frac{-242}{120}$ ... then determine the value of a_6 and whether the sequence converges or diverges.

$$\{a_n\} = \frac{(-1)^n 3^n - 1}{n!}$$

$$a_6 = \frac{(-1)^6 3^6 - 1}{6!} = \frac{728}{720}$$

$n!$ grows much faster than 3^n

$$\frac{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot \ldots}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \ldots}$$

therefore, the sequence converges to zero