9.4 Limit Comparison Test

Pg. 616 #'s 13-21 odd, 23-30 (omit 28)

13) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ compare to $\frac{n}{n^2} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{n}{n^2} \cdot \frac{n^2+1}{n} = \frac{n^2+1}{n^2} = \frac{\infty}{\infty}$

l'Hop

$\lim_{n\to\infty} \frac{2n}{2n} = \frac{\infty}{\infty}$

l'Hop

$\lim_{n\to\infty} \frac{2}{2} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series ($p \le 1$),

so $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by limit comp.

15) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ compare to $\frac{1}{\sqrt{n^2}} = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n^2}} \cdot \frac{\sqrt{n^2+1}}{1} = \frac{\infty}{\infty}$

$\lim_{n\to\infty} \sqrt{\frac{n^2+1}{n^2}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} = \sqrt{1+\frac{1}{\infty}} = 1$

$\sum_{n=0}^{\infty} \frac{1}{n}$ diverges by p-series ($p \le 1$), so

$\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges by limit comparison

17) $\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$ compare to $\frac{n^2}{n^5} = \frac{1}{n^3}$

$\lim_{n\to\infty} \frac{n^2}{n^5} \cdot \frac{3n^5+2n+1}{2n^2-1} = \frac{3n^7+2n^3+n^2}{2n^7-n^5} = \frac{3}{2}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series ($p>1$),

so $\sum_{n=1}^{\infty} \frac{2n^2-1}{3n^5+2n+1}$ converges by limit comp.

19) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ compare to $\frac{1}{n\sqrt{n^2}} = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{1}{n\sqrt{n^2}} \cdot \frac{n\sqrt{n^2+1}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^2}}$

$\lim_{n\to\infty} \sqrt{\frac{n^2+1}{n^2}} = \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} = \sqrt{1+0} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series ($p>1$),

so $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+1}}$ converges by limit comp.

21) $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}$ compare to $\frac{1}{n}$

$\lim_{n\to\infty} \frac{1}{n} \cdot \frac{n^k+1}{n^{k-1}} = \lim_{n\to\infty} \frac{n^k+1}{n^k} = 1$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series, so $\sum_{n=1}^{\infty} \frac{n^{k-1}}{n^k+1}$ diverges by limit comparison.
($p \le 1$)

23) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n} = \frac{1}{n^{2/3}}$

$p = 2/3 < 1$

The series diverges by p-series test.

24) $\sum_{n=0}^{\infty} 5\left(-\frac{4}{3}\right)^n$

$r = -\frac{4}{3}$ $\quad \left|-\frac{4}{3}\right| \geq 1$

The series diverges by Geometric series test.

25) $\sum_{n=1}^{\infty} \frac{1}{5^n+1}$ compare to $\frac{1}{5^n}$

$\frac{1}{5^n} > \frac{1}{5^n+1}$

$\sum_{n=0}^{\infty} \frac{1}{5^n}$ converges by Geometric series test $(r<1)$, so $\sum_{n=0}^{\infty} \frac{1}{5^n+1}$ converges by direct comparison.

26) $\sum_{n=2}^{\infty} \frac{1}{n^3-8}$ compare to $\frac{1}{n^3}$

$\lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{n^3-8}{1} = \frac{n^3-8}{n^3} = 1$

$\sum_{n=2}^{\infty} \frac{1}{n^3}$ converges by p-series $(p>1)$,

so $\sum_{n=2}^{\infty} \frac{1}{n^3-8}$ converges by limit comp.

27) $\sum_{n=1}^{\infty} \frac{2n}{3n-2}$

$\lim_{n\to\infty} \frac{2n}{3n-2} = \frac{2}{3} \neq 0$

the series diverges by the n^{th} term test

29) $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$

$f(x) = \frac{x}{(x^2+1)^2}$

$\lim_{a\to\infty} \int_1^a \frac{x}{(x^2+1)^2}\,dx$ $\quad u = x^2+1 \quad u(a) = a^2+1$

$du = 2x\,dx \quad u(1) = 2$

$\frac{1}{2}du = x\,dx$

$\lim_{a\to\infty} \int_2^{a^2+1} \frac{1}{2}u^{-2}\,du$

$\lim_{a\to\infty} \left[-\frac{1}{2}u^{-1}\right]_2^{a^2+1}$

$\lim_{a\to\infty} \left[\frac{-1}{2(a^2+1)} - \frac{-1}{2(2)}\right] = 0 + \frac{1}{4} = \frac{1}{4}$

$\int_1^{\infty} \frac{x}{(x^2+1)^2}\,dx$ converges so $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges by integral test

30) $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ compare to $\frac{1}{n^2}$

$\frac{1}{n^2} > \frac{1}{n(n+3)}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series $(p>2)$,

so $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ converges by direct comp.