Calculus Section 9.4 Limit Comparison Test

Homework: page 616 #'s 13 – 21 odd, 23 – 30 (omit 28)

-Use the limit comparison test to determine convergence or divergence.

Some series closely resemble others but you are unable to apply the Direct Comparison Test. If this is the case, there is a second comparison test called the Limit Comparison Test.

$\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$ is a good example where direct comparison will not work but limit comparison will.

Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is *finite and positive.* Then the two series $\sum a_n$ and $\sum b_n$ either both converge or both diverge. (further clarification: if L is finite and positive, then L cannot equal zero and L cannot equal infinity)

Choosing what to compare:

$$\sum_{n=1}^{\infty} \frac{1}{3n^2-4n+5} \qquad \frac{1}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}} \qquad \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{3n-2}} \qquad \frac{n^2}{\sqrt{n}} = \frac{1}{n^{3/2}}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n-1} \qquad \frac{1}{2^n}$$

Example) Determine the convergence or divergence of the following series

1) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ compare to $\frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n^2}}{\frac{\sqrt{n}}{n^2+1}} = \lim_{n\to\infty} \frac{\sqrt{n}}{n^2} \cdot \frac{n^2+1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{n^2+1}{n^2} = \frac{\infty}{\infty}$$

l'Hop

$$\lim_{n\to\infty} \frac{2n}{2n} = \frac{\infty}{\infty}$$

l'Hop

$$\lim_{n\to\infty} \frac{2}{2} = 1$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-series test $(p>1)$,

so, $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ converges by limit comparison.

2) $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ compare to $\frac{n2^n}{n^3} = \frac{2^n}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{n2^n}{n^3}}{\frac{n2^n}{4n^3+1}} = \lim_{n\to\infty} \frac{n2^n}{n^3} \cdot \frac{4n^3+1}{n2^n}$$

$$\lim_{n\to\infty} \frac{4n^3+1}{n^3} = 4$$

$\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ diverges by the n^{th} term test,

so $\sum_{n=1}^{\infty} \frac{n2^n}{4n^3+1}$ diverges by limit comparison.

3) $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-2}}$ compare to $\frac{1}{\sqrt[3]{n^2}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \cdot \frac{\sqrt[3]{n^2-2}}{1}$$

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2-2}}{\sqrt[3]{n^2}}$$

$$\lim_{n\to\infty} \sqrt[3]{\frac{n^2-2}{n^2}}$$

$$\lim_{n\to\infty} \sqrt[3]{1-\frac{2}{n^2}} = \sqrt[3]{1-0} = 1$$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ diverges by p-series $(p \leq 1)$,

so $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-2}}$ diverges by limit comparison.