Calculus Section 9.6 Ratio Test

-Use the ratio test to determine convergence or divergence

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The ratio test is a test that determines whether a function converges absolutely.

Ratio Test

Let $\sum a_n$ be a series with nonzero terms.

1) $\sum a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$

3) The Ratio Test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

The Ratio Test is particularly useful for series that converge rapidly (i.e. factorials or exponentials).

Example) Determine Convergence or Divergence

1) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$$

$$\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}$$

$$\lim_{n\to\infty} \frac{2\cdot 2^n}{(n+1)n!} \cdot \frac{n!}{2^n}$$

$$\lim_{n\to\infty} \frac{2}{n+1} = 0 < 1$$

The series converges by the ratio test.

2) $\sum_{n=0}^{\infty} \frac{n^2 2^{n+1}}{3^n}$

$$\lim_{n\to\infty} \frac{(n+1)^2\, 2^{n+2}}{3^{n+1}} \cdot \frac{3^n}{n^2 \cdot 2^{n+1}}$$

$$\lim_{n\to\infty} \frac{(n+1)^2 \cdot 2}{3n^2}$$

$$\lim_{n\to\infty} \frac{2n^2+4n+2}{3n^2} = \frac{2}{3} < 1$$

The series converges by the ratio test.

3) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)(n+1)^n}{(n+1)\,n!} \cdot \frac{n!}{n^n}$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$$

The series diverges by the ratio test.

$$y = \left(1+\frac{1}{n}\right)^n$$

$$\ln y = n\ln\left(1+\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{\ln(1+\frac{1}{n})}{1/n} = \frac{0}{0}$$

l'Hop

$$\lim_{n\to\infty} \frac{\left(\frac{1}{1+1/n}\right)(-1/n^2)}{(-1/n^2)}$$

$$\lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$$

$$\ln y = 1$$

$$e^{\ln y} = e^1$$

$$y = e$$

4) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

$$\lim_{n\to\infty} \frac{\sqrt{n+1}}{n+2} \cdot \frac{n+1}{\sqrt{n}}$$

$$\lim_{n\to\infty} \sqrt{\frac{n+1}{n}} \cdot \frac{n+1}{n+2}$$

$$\lim_{n\to\infty} \sqrt{1+\frac{1}{n}} \cdot \frac{n+1}{n+2} = \sqrt{1} + 1 = 1$$

The ratio test is inconclusive.

1) $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ ✓

2) $\frac{\sqrt{n+1}}{n+2} \leq \frac{\sqrt{n}}{n+1}$ ✓

The series converges by the alternating series test.