Calculus Section 9.7 Approximations and Lagrange Error Bound

-Use a Taylor or Maclaurin polynomial to approximate a function
-Find the Lagrange error for a polynomial approximation

Homework: Lagrange Error Bound Worksheet

One major application of Taylor and Maclaurin polynomials is approximating the value of a function.

Example)
Use a 4th degree Taylor polynomial centered at 0 to approximate f(0.2) for the function f(x) = e^x.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$f(0.2) \approx 1 + .2 + \frac{(.2)^2}{2} + \frac{(.2)^3}{3!} + \frac{(.2)^4}{4!}$$

$$f(0.2) \approx 1.2214$$

There is error associated with any approximation. We use the Alternating Series Remainder Theorem to find the maximum possible error for a convergent alternating series. But what if the series does not alternate?

Lagrange Error Bound
Let $P_n(x)$ be the nth degree Taylor polynomial for f(x) centered about x = c.

$$\text{Error} = |P_n(x_0) - f(x_0)| \le \left|\frac{f^{(n+1)}(z)}{(n+1)!}(x_0 - c)^{n+1}\right|$$

Where x_0 is where the function is being approximated and $f^{(n+1)}(z)$ is the maximum value of the n + 1 term between x_0 and c.

We do not need to find the value of z. We just need to find a reasonable, safe bound for $f^{(n+1)}(z)$.

Example)
Use Lagrange error to prove that $|f(0.2) - P(0.2)| < \frac{1}{1000}$ for the previous example.

Next term: $\frac{x^5}{5!}$

Error: $\frac{(max)}{5!}(.2-0)^5$

What is the maximum value of the fifth derivative of $f(x)=e^x$ on the interval $(0, .2)$?

It would be $e^{.2}$, but I don't know the number offhand (and would defeat the purpose of this problem). Choose a safe value: $e^1 \approx 2.7 < 3$.

$$\frac{3}{5!}(.2)^5 \rightarrow \frac{3}{5!}\left(\frac{1}{5}\right)^5 \rightarrow \frac{3}{5^5 \cdot 5!}$$

$$\frac{3}{3125 \cdot 120} \rightarrow \frac{1}{3125 \cdot 40} < \frac{1}{1000}$$

Even a conservative over-estimate of the max value yields an error < $\frac{1}{1000}$. →

Example)

Let f be the function given by $f(x)=\sin\left(5x+\frac{\pi}{3}\right)$ and let P(x) be the third-degree Taylor polynomial for f about x = 0. Find P(x). Then use the Lagrange error bound to show that $\left|f\left(\frac{1}{15}\right)-P\left(\frac{1}{15}\right)\right|<\frac{1}{1200}$.

$f(x)=\sin\left(5x+\frac{\pi}{3}\right)$ $\quad f(0)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

$f'(x)=5\cos\left(5x+\frac{\pi}{3}\right)$ $\quad f'(0)=5\cos\left(\frac{\pi}{3}\right)=\frac{5}{2}$

$f''(x)=-25\sin\left(5x+\frac{\pi}{3}\right)$ $\quad f''(0)=-25\sin\left(\frac{\pi}{3}\right)=\frac{-25\sqrt{3}}{2}$

$f'''(x)=-125\cos\left(5x-\frac{\pi}{3}\right)$ $\quad f'''(0)=-125\cos\left(\frac{\pi}{3}\right)=\frac{-125}{2}$

$$P(x)=\frac{\sqrt{3}}{2}+\frac{5}{2}x-\frac{\frac{25\sqrt{3}}{2}x^2}{2!}-\frac{\frac{125}{2}x^3}{3!}$$

Error: $\frac{(\max)}{4!}\left(\frac{1}{15}-0\right)^4$

$f''''(x)=625\sin\left(5x-\frac{\pi}{3}\right)$

max of $f''''(x)$ is <625

$$\frac{625}{4!}\left(\frac{1}{15}\right)^4\rightarrow\frac{625}{15^4\cdot 24}\rightarrow\frac{625}{(15^2\cdot 3)(15^2\cdot 8)}$$

$$\frac{625}{(725)(1800)}<\frac{1}{1800}<\frac{1}{1200}$$

$\frac{625}{725}<1$

Example)

x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
1	10	5	6	-2
2	6	4	-7	8
3	-3	-2	9/4	3

centered at 2

Write a third-degree Taylor Polynomial for f about x = 2, and use it to approximate $f(2.3)$. If the fourth derivative of f satisfies the inequality $\left|f^{(4)}(x)\right|\leq 9$ for all x in the interval, use the Lagrange error bound to find an interval [a,b] such that a ≤ $f(2.3)$ ≤ b.

$$P(x)=6+4(x-2)-\frac{7(x-2)^2}{2}+\frac{8(x-2)^3}{3!}$$

Error: $\frac{(\max)}{4!}(2.3-2)^4$

$$f(2.3)\approx 6+4(2.3-2)-\frac{7(2.3-2)^2}{2}+\frac{8(2.3-2)^3}{3!}$$

$\frac{9}{4!}(.3)^4$

$f(2.3)\approx 6.921$

Max Error = .0030375

$6.918\leq f(2.3)\leq 6.924$