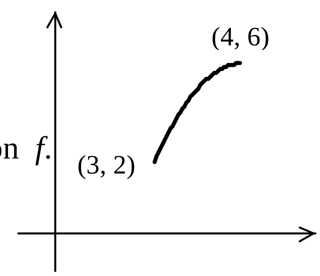
Lagrange Error Bound Worksheet

1. Let f be a function that has derivatives of all orders on the interval $(-1, 1)$. Assume $f(0) = 1$,
   $f'(0) = \frac{1}{2}$, $f''(0) = -\frac{1}{4}$, $f'''(0) = \frac{3}{8}$, and $\left|f^{(4)}(x)\right| \le 6$ for all x in the interval (0, 1).
   (a) Find the third-degree Taylor polynomial about $x = 0$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(0.5)$.
   (c) What is the maximum possible error for the approximation made in part (b)?

2. Let f be the function defined by $f(x) = \sqrt{x}$.
   (a) Find the second-degree Taylor polynomial about $x = 4$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(4.2)$.
   (c) Find a bound on the error for the approximation in part (b).

3. (Calc) Let f be a function that has derivatives of all orders. Assume $f(3) = 1$,
   $f'(3) = \frac{1}{2}$, $f''(3) = -\frac{1}{4}$, $f'''(3) = \frac{3}{8}$, and the graph of $f^{(4)}(x)$ on [3, 4] is shown on the right. The graph of $f^{(4)}(x)$ is increasing on [3, 4].

   Graph of $f^{(4)}(x)$

   (a) Find the third-degree Taylor polynomial about $x = 3$ for the function f.
   (b) Use your answer to part (a) to estimate the value of $f(3.7)$.
   (c) Use information from the graph of $y = f^{(4)}(x)$ to show that $\left|f(3.7) - P(3.7)\right| < 0.08$.
   (d) Could $f(3.7)$ equal 1.283? Show why or why not.

4. Estimate the error that results when $\sin x$ is replaced by $x - \frac{1}{6}x^3$ for $|x| < 0.2$.
   Show your reasoning.

5. Use series to find an estimate for $\int_0^1 e^{-x^2}\,dx$ that is accurate to three decimal places. Justify your answer.

6. Suppose a function f is approximated with a fourth-degree Taylor polynomial about $x = 1$. If the maximum value of the fifth derivative between $x = 1$ and $x = 3$ is 0.01, that is, $\left|f^{(5)}(x)\right| < 0.01$, then the maximum error incurred using this approximation to compute $f(3)$ is

(A) 0.054 (B) 0.0054 (C) 0.26667 (D) 0.02667 (E) 0.00267

7. The maximum error incurred by approximating the sum of the series $1 - \frac{1}{2!} + \frac{2}{3!} - \frac{3}{4!} + \frac{4}{5!} - \ldots$ by the sum of the first six terms is

(A) 0.001190 (B) 0.006944 (C) 0.33333 (D) 0.125000 (E) None of these

8. The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \frac{(-1)^n n!}{2^n (n+2)}$ and $f(5) = \frac{1}{2}$. Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with an error less than $\frac{1}{1000}$.

Answers to Worksheet on Series and Error

1. (a) $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$

   (b) $\frac{157}{128}=1.22656$

   (c) The error is at most $\frac{1}{64}=0.015625$

2. (a) $2+\frac{x-4}{4}-\frac{(x-4)^2}{64}$

   (b) 2.049375

   (c) The maximum value of the third derivative $f'''(x)=\frac{3}{8x^{5/2}}$ on [4, 4.2] occurs at $x = 4$ and

   is $\frac{3}{256}$. Then $\left|R_2(x)\right|\le\frac{3/256}{3!}(0.2)^3=1.5625\times10^{-5}$

3. (a) $1+\frac{x-3}{2}-\frac{(x-3)^2}{4\cdot2!}+\frac{3(x-3)^2}{8\cdot3!}$

   (b) 1.310

   (c) Since $f^{(4)}(x)$ is increasing on [3, 4], $f^{(4)}(x)<6$ on [3, 3.7] so

   $\left|\text{Error}\right|<\left|\frac{6(3.7-3)^4}{4!}\right|=0.060<0.08$.

   (d) Yes, $1.250\le f(3.7)\le1.370$ so $f(3.7)$ could equal 1.283.

4. (a) By the Lagrange Error Bound, since the derivatives of sin *x* are

   $\cos x$, $-\sin x$, and $-\cos x$, and $\left|\cos x\right|\le1$ and $\left|\sin x\right|\le1$, then

   $\left|R_3(x)\right|\le\frac{(0.2)^4}{4!}=0.000067$.

5. Because this is an alternating series whose terms decrease in value, we can truncate after 6 terms and have an error correct to three decimal places.

   $\int_0^1 e^{-x^2}dx\approx1-\frac{1}{3}+\frac{1}{5(2!)}-\frac{1}{7(3!)}+\frac{1}{9(4!)}-\frac{1}{11(5!)}=0.746729$

6. E
7. A
8. $f(6)=\frac{1}{2}-\frac{1}{6}+\frac{1}{16}-\frac{1}{40}+\frac{1}{96}-\frac{1}{224}+\frac{1}{512}-\frac{1}{1152}+\ldots$

   This is an alternating series whose terms are decreasing in size so the error involved in approximating $f(6)$ with the sixth-degree Taylor polynomial is less in magnitude than the seventh-degree term.

   $\left|\text{Error}\right|<\frac{1}{1152}<\frac{1}{1000}$ by the Alternating Series Remainder