Calculus Section 9.9 Geometric Power Series

-Find a geometric power series that represents a function.
-Construct a power series using series operations.

Homework: page 662 #'s 5, 7, 9, 11, 12, 35, 36

The final type of series involves functions written in the form: $f(x) = \frac{1}{1-x}$. These series resemble the sum of a Geometric series: $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots = \frac{a}{1-r}$. We can infer that the power series expansion of $f(x) = \frac{1}{1-x}$ is $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} 1 \cdot x^n$

Examples) Find a Power Series and Interval of Convergence for Each Function

$f(x) = \frac{1}{1-2x}$ c = 0

$a = 1 \quad r = 2x$

$$\boxed{\sum_{n=0}^{\infty} 1 \cdot (2x)^n}$$

$|2x| < 1$

$-1 < 2x < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

$\boxed{\text{Interval: } \left(\frac{-1}{2}, \frac{1}{2}\right)}$

You do not have to check the endpoints of a geometric series because the divergent condition is $|r| \geq 1$. Thus, the endpoints always diverge.

$f(x) = \frac{1}{4+x}$ c = 2

$f(x) = \frac{1}{4+2+(x-2)}$

$f(x) = \frac{1}{6+(x-2)}$

$f(x) = \frac{1/6}{1+\frac{1}{6}(x-2)}$

$f(x) = \frac{1/6}{1-\frac{-1}{6}(x-2)} \qquad a = 1/6 \quad r = \frac{-1}{6}(x-2)$

$$\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)\left(\frac{-(x-2)}{6}\right)^n$$

$$\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n (x-2)^2}{6^{n+1}}}$$

$\left|\frac{-1}{6}(x-2)\right| < 1$

$-1 < \frac{x-2}{6} < 1$

$-6 < x-2 < 6$

$-4 < x < 8 \qquad \boxed{\text{Interval: } (-4, 8)}$

$f(x) = \frac{3}{3-2x}$, c = 1

$$f(x) = \frac{3}{3-2-2(x-1)}$$

$$f(x) = \frac{3}{1-2(x-1)}$$

$a = 3 \quad r = 2(x-1)$

$$\boxed{\sum_{n=0}^{\infty} 3(2(x-1))^n}$$

$$|2(x-1)| < 1$$

$$-1 < 2(x-1) < 1$$

$$-\frac{1}{2} < x-1 < \frac{1}{2}$$

$$\frac{1}{2} < x < \frac{3}{2}$$

$$\boxed{\text{Interval: } \left(\frac{1}{2}, \frac{3}{2}\right)}$$

Find the power series representation of the function $f(x) = \frac{1}{(1-x)^2}$.

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right]$$

$$\frac{d}{dx}\left[\frac{1}{1-x}\right] = \frac{(1-x)(0) - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} 1 \cdot x^n$$

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\sum_{n=0}^{\infty} 1 \cdot x^n\right]$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}}$$

Find the power series representation for f(x) = arctan(x).

$$\arctan x = \int \frac{1}{1+x^2}\,dx \quad a=1, \; r=-x^2$$

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} 1\cdot(-x^2)^n = \sum_{n=0}^{\infty} (-1)^n (x)^{2n}$$

$$\arctan x = \int\left(\sum_{n=0}^{\infty} (-1)^n x^{2n}\right)dx$$

$$\boxed{\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$