AP Questions Tay/Mac #2

2012 AP Test

The function g has derivatives of all orders, and the Maclaurin series for g is
$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $$\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$$

$$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$$

$x^2 < 1 \Rightarrow -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $$\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$$

2 :
- 1 : uses the third term as an error bound
- 1 : error bound

(c) $$g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$$

2 :
- 1 : first three terms
- 1 : general term

2010 Test

$$f(x)=\begin{cases}\dfrac{\cos x-1}{x^2} & \text{for } x\neq 0\\ -\dfrac{1}{2} & \text{for } x=0\end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x)=1+\int_0^x f(t)\,dt$.

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x=0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x=0$.

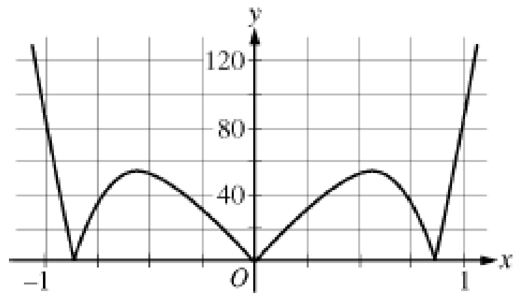
(b) Use the Taylor series for f about $x=0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x=0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x=0$.

(d) The Taylor series for g about $x=0$, evaluated at $x=1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x=0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

(a) $\cos(x)=1-\frac{x^2}{2}+\frac{x^4}{4!}-\cdots+(-1)^n\frac{x^{2n}}{(2n)!}+\cdots$ $f(x)=-\frac{1}{2}+\frac{x^2}{4!}-\frac{x^4}{6!}+\cdots+(-1)^{n+1}\frac{x^{2n}}{(2n+2)!}+\cdots$	3 : { 1 : terms for $\cos x$ 2 : terms for f 1 : first three terms 1 : general term }
(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x=0$, so $f'(0)=0$. $\frac{f''(0)}{2!}=\frac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x=0$, so $f''(0)=\frac{1}{12}$. Therefore, by the Second Derivative Test, f has a relative minimum at $x=0$.	2 : { 1 : determines $f'(0)$ 1 : answer with reason }
(c) $P_5(x)=1-\frac{x}{2}+\frac{x^3}{3\cdot 4!}-\frac{x^5}{5\cdot 6!}$	2 : { 1 : two correct terms 1 : remaining terms }
(d) $g(1)\approx 1-\frac{1}{2}+\frac{1}{3\cdot 4!}=\frac{37}{72}$ Since the Taylor series for g about $x=0$ evaluated at $x=1$ is alternating and the terms decrease in absolute value to 0, we know $\left\lvert g(1)-\frac{37}{72}\right\rvert<\frac{1}{5\cdot 6!}<\frac{1}{6!}$.	2 : { 1 : estimate 1 : explanation }

2011 Test



Graph of $y = \left|f^{(5)}(x)\right|$

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$	3 : { 1 : series for $\sin x$ 2 : series for $\sin\left(x^2\right)$
(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$	3 : { 1 : series for $\cos x$ 2 : series for $f(x)$
(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.	1 : answer
(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$. Therefore $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}$.	2 : { 1 : form of the error bound 1 : analysis