

AP Questions Tay/Mac #3

2009 Test

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + \frac{x^n}{n!} + \dots$. The continuous function f is defined

by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

- (a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.
 (b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.
 (c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).
 (d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \dots + \frac{(x-1)^{2n}}{n!} + \dots$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \dots + \frac{(x-1)^{2n}}{(n+1)!} + \dots$

2 : $\begin{cases} 1 : \text{first four terms} \\ 1 : \text{general term} \end{cases}$

(c) $\lim_{n \rightarrow \infty} \left| \frac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!}{(n+2)!} (x-1)^2 = \lim_{n \rightarrow \infty} \frac{(x-1)^2}{n+2} = 0$

3 : $\begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{answer} \end{cases}$

Therefore, the interval of convergence is $(-\infty, \infty)$.

(d) $f''(x) = 1 + \frac{4 \cdot 3}{6}(x-1)^2 + \frac{6 \cdot 5}{24}(x-1)^4 + \dots$
 $+ \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \dots$

2 : $\begin{cases} 1 : f''(x) \\ 1 : \text{answer} \end{cases}$

Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x .
 Therefore, the graph of f has no points of inflection.

2007 Test

Let f be the function given by $f(x) = e^{-x^2}$.

- (a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.
- (b) Use your answer to part (a) to find $\lim_{x \rightarrow 0} \frac{1 - x^2 - f(x)}{x^4}$.
- (c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2} dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2} dt$.
- (d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2} dt$ by less than $\frac{1}{200}$.

$$\begin{aligned} \text{(a)} \quad e^{-x^2} &= 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots \\ &= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots \end{aligned}$$

3 : $\begin{cases} 1 : \text{two of } 1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6} \\ 1 : \text{remaining terms} \\ 1 : \text{general term} \end{cases}$

$$\begin{aligned} \text{(b)} \quad \frac{1 - x^2 - f(x)}{x^4} &= -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!} \\ \text{Thus, } \lim_{x \rightarrow 0} \left(\frac{1 - x^2 - f(x)}{x^4} \right) &= -\frac{1}{2}. \end{aligned}$$

1 : answer

$$\begin{aligned} \text{(c)} \quad \int_0^x e^{-t^2} dt &= \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots \right) dt \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \end{aligned}$$

3 : $\begin{cases} 1 : \text{two terms} \\ 1 : \text{remaining terms} \\ 1 : \text{estimate} \end{cases}$

Using the first two terms of this series, we estimate that

$$\int_0^{1/2} e^{-t^2} dt = \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}.$$

$$\text{(d)} \quad \left| \int_0^{1/2} e^{-t^2} dt - \frac{11}{24} \right| < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200}, \text{ since}$$

$\int_0^{1/2} e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)}$, which is an alternating series with individual terms that decrease in absolute value to 0.

2 : $\begin{cases} 1 : \text{uses the third term as the error bound} \\ 1 : \text{explanation} \end{cases}$

2006 Test

The function f is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \dots + \frac{(-1)^n nx^n}{n+1} + \dots$$

for all real numbers x for which the series converges. The function g is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots + \frac{(-1)^n x^n}{(2n)!} + \dots$$

for all real numbers x for which the series converges.

- (a) Find the interval of convergence of the power series for f . Justify your answer.
 (b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether y has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

$$(a) \left| \frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n} \right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$$

$$\lim_{n \rightarrow \infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \dots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

$$(b) f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \dots \text{ and } f'(0) = -\frac{1}{2}.$$

$$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \dots \text{ and } g'(0) = -\frac{1}{2}.$$

$$y'(0) = f'(0) - g'(0) = 0$$

$$f''(0) = \frac{4}{3} \text{ and } g''(0) = \frac{2}{4!} = \frac{1}{12}.$$

$$\text{Thus, } y''(0) = \frac{4}{3} - \frac{1}{12} > 0.$$

Since $y'(0) = 0$ and $y''(0) > 0$, y has a relative minimum at $x = 0$.

- 5 : $\left\{ \begin{array}{l} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{identifies radius of convergence} \\ 1 : \text{considers both endpoints} \\ 1 : \text{analysis/conclusion for both endpoints} \end{array} \right.$

- 4 : $\left\{ \begin{array}{l} 1 : y'(0) \\ 1 : y''(0) \\ 1 : \text{conclusion} \\ 1 : \text{reasoning} \end{array} \right.$