AP Calculus AB Review Transcendentals, Diff Eqs., Area/Volume Name: Answer Key

Determine the derivative of the following functions.

1) $f(x) = \ln(x^4 + 8)$

$\boxed{\frac{1}{x^4+8} * (4x^3)}$

2) $f(x) = e^{\sec x}$

$\boxed{e^{\sec x} * (\sec x \tan x)}$

3) $g(x) = \tan(e^{2x})$

$\boxed{\sec^2(e^{2x}) * e^{2x} * 2}$

4) $\int_{\ln(2x)}^{1} f(t)dt$

$-\int_{1}^{\ln 2x} f(t)dt$

$\boxed{-f(\ln(2x)) * \frac{1}{2x} * 2}$

5) $y = \arctan(3x - 1)$

$\boxed{\frac{3}{1+(3x-1)^2}}$

6) $f(x) = 2^{4x} - \cot(3x)$

$\boxed{(\ln 2) 2^{4x} * 4 + 3\csc^2(3x)}$

Evaluate the following integrals.

7) $\int \frac{\sec^2 x}{\tan x} dx$

$u = \tan x$

$du = \sec^2 x\, dx$

$\int \frac{1}{u} du$

$\ln|u| + C$

$\boxed{\ln|\tan x| + C}$

8) $\int \frac{1}{x \ln x} dx$

$u = \ln x$

$du = \frac{1}{x} dx$

$\int \frac{1}{u} du$

$\ln|u| + C$

$\boxed{\ln|\ln x| + C}$

9) $\int e^{5x} dx$

$u = 5x$

$du = 5dx$

$\frac{1}{5} du = dx$

$\frac{1}{5}\int e^u du$

$\frac{1}{5} e^u + C$

$\boxed{\frac{1}{5} e^{5x} + C}$

10) $\int \frac{e^x}{1+e^x} dx$

$u = 1 + e^x$

$du = e^x dx$

$\int \frac{1}{u} du$

$\ln|u| + C$

$\boxed{\ln|1+e^x| + C}$

11) $\int \frac{e^x}{1+e^{2x}} dx$

$a = 1$ $u = e^x$

$du = e^x dx$

$\int \frac{du}{a^2+u^2}$

$\frac{1}{1}\arctan\left(\frac{e^x}{1}\right) + C$

$\boxed{\arctan(e^x) + C}$

12) $\int \tan(4x)\, dx$

$u = 4x$

$du = 4dx$

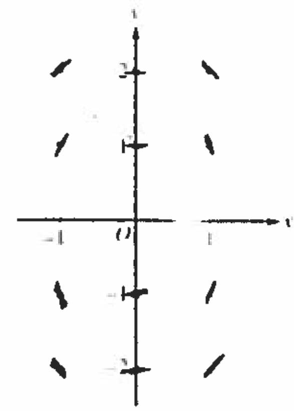
$\frac{1}{4} du = dx$

$\frac{1}{4}\int \tan(u) du$

$\boxed{-\frac{1}{4}\ln|\cos 4x| + C}$

13) Consider the differential equation $\frac{dy}{dx} = -\frac{2x}{y}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated.



(b) Let y= f(x) be the particular solution to the differential equation with the initial condition f(1) = -1. Write an equation for the line tangent to the graph of *f* at (1, -1) and use it to approximate f(1.1).

$\frac{dy}{dx}$ @ $(1,-1) = \frac{-2(1)}{-1} = 2$

$\boxed{y+1 = 2(x-1)}$

$y = 2(1.1-1) - 1$

$y = 2(.1) - 1$

$y = .2 - 1$

$y = -.8$

$\boxed{f(1.1) \approx -.8}$

(c) Is the approximation for f(1.1) found in part (b) an underestimate or overestimate? Justify your reasoning.

$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{-2x}{y}\right]$

$\frac{d^2y}{dx^2} = \frac{y(-2) - (-2x)\frac{dy}{dx}}{y^2}$

$\frac{d^2y}{dx^2}$ @ $(1,-1) = \frac{(-1)(-2) - (-2)(2)}{(-1)^2} = \frac{2+4}{1} = 6$

The graph is concave up, so f(1.1) is an underestimate.

(d) Find the particular solution y = f(x) to the given differential equation with the initial condition f(1) = -1.

$\int y\,dy = \int -2x\,dx$

$\frac{1}{2}y^2 = -x^2 + C$

$\frac{1}{2}(-1)^2 = -(1)^2 + C$

$\frac{1}{2} = -1 + C$

$\frac{3}{2} = C$

$\frac{1}{2}y^2 = -x^2 + 1.5$

$y^2 = -2x^2 + 3$

$\boxed{y = \pm\sqrt{3-2x^2}}$

14) Find the particular solution y = f(x) to the differential equation $\frac{dy}{dx} = x^2(y-1)$ with the initial condition f(0) = 3.

$$\frac{1}{y-1}\,dy = x^2\,dx$$

$$\ln|y-1| = \frac{1}{3}x^3 + C$$

$$y-1 = Ce^{\frac{1}{3}x^3}$$

$$3-1 = Ce^{\frac{1}{3}(0)^3}$$

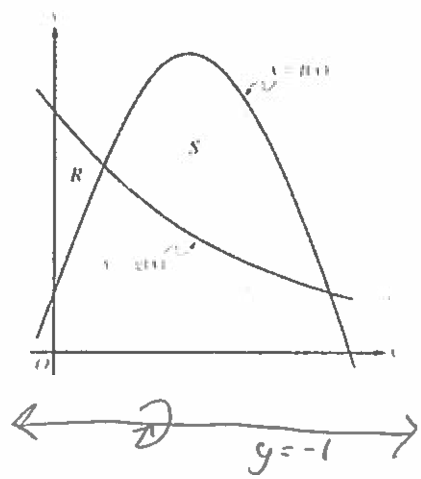
$$2 = Ce^0$$

$$2 = C$$

$$y-1 = 2e^{\frac{1}{3}x^3}$$

$$\boxed{y = 2e^{\frac{1}{3}x^3} + 1}$$

15) Let *f* and *g* be the functions given by f(x) = $\frac{1}{4} + \sin(\pi x)$ and g(x) = 4^{-x}. Let R be the shaded region in the first quadrant enclosed by the y-axis and the graphs of *f* and *g*, and let S be the shaded region in the first quadrant enclosed by the graphs of *f* and *g*, as shown in the figure below.



(a) Find the area of R.

$$A = \int_0^{.178213} \left(4^{-x} - \left(\frac{1}{4} + \sin(\pi x)\right)\right)dx$$

$$\boxed{A = .065}$$

(b) Find the area of S.

$$A = \int_{.178213}^{1} \left(\left(\frac{1}{4} + \sin(\pi x)\right) - 4^{-x}\right)dx$$

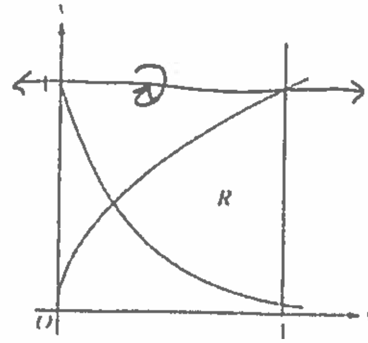
$$\boxed{A = .410}$$

(c) Find the volume of the solid generated when S is revolved about the horizontal line y = -1.

$$V = \pi\int_{.178213}^{1} \left(\frac{1}{4} + \sin(\pi x) + 1\right)^2 dx - \pi\int_{.178213}^{1} \left(4^{-x} + 1\right)^2 dx$$

$$\boxed{V = 4.559}$$

16) Let R be the shaded region bounded by the graphs of $y = \sqrt{x}$ and $y = e^{-3x}$ and the vertical line x = 1, as shown in the figure below.



(a) Find the area of R.

$$A = \int_{.238734}^{1} \left(\sqrt{x} - e^{-3x}\right) dx$$

$$\boxed{A = .443}$$

(b) Find the volume of the solid generated when R is revolved about the horizontal line y = 1.

$$V = \pi \int_{.238734}^{1} \left(1 - e^{-3x}\right)^2 dx - \pi \int_{.238734}^{1} \left(1 - \sqrt{x}\right)^2 dx$$

$$\boxed{V = 1.424}$$

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a rectangle whose height is 5 times the length of its base in region R. Find the volume of this solid.

$$V = \int_{.238734}^{1} \left(\sqrt{x} - e^{-3x}\right)\left(5\left(\sqrt{x} - e^{-3x}\right)\right) dx$$

$$\boxed{V = 1.554}$$

17) The radius r of a sphere is increasing at a constant rate of 0.04 centimeters per second. (Note: The volume of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

(a) At the time when the radius of the sphere is 10 centimeters, what is the rate of increase of its volume?

$$\frac{d}{dt}\left[V = \frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi (10)^2 (0.04)$$

$$\boxed{\frac{dV}{dt} = 16\pi \text{ cm}^3/\text{s}}$$

(b) At the time when the volume of the sphere is 36π cubic centimeters, what is the rate of increase of the area of a cross section through the center of the sphere?

$$V = \frac{4}{3}\pi r^3$$

$$36\pi = \frac{4}{3}\pi r^3$$

$$27 = r^3$$

$$3 = r$$

$$\frac{d}{dt}\left[A = \pi r^2\right]$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi (3)(0.04)$$

$$\boxed{\frac{dA}{dt} = .24\pi \text{ cm}^2/\text{s}}$$